Persistence of small-scale anisotropies and anomalous scaling in a model of magnetohydrodynamics turbulence

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The problem of anomalous scaling in magnetohydrodynamics turbulence is considered within the framework of the kinematic approximation, in the presence of a large-scale background magnetic field. The velocity field is Gaussian, δ-correlated in time, and scales with a positive exponent ξ. Explicit inertial-range expressions for the magnetic correlation functions are obtained; they are represented by superpositions of power laws with nonuniversal amplitudes and universal (independent of the anisotropy and forcing) anomalous exponents. The complete set of anomalous exponents for the pair correlation function is found nonperturbatively, in any space dimension d, using the zero-mode technique. For higher-order correlation functions, the anomalous exponents are calculated to $O(\xi)$ using the renormalization group. The exponents exhibit a hierarchy related to the degree of anisotropy; the leading contributions to the even correlation functions are given by the exponents from the isotropic shell, in agreement with the idea of restored small-scale isotropy. Conversely, the small-scale anisotropy reveals itself in the odd correlation functions: the skewness factor is slowly decreasing going down to small scales and higher odd dimensionless ratios (hyperskewness, etc.) dramatically increase, thus diverging in the $r\rightarrow 0$ limit.

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I. INTRODUCTION

In cosmical objects, small-scale evolution of the magnetic field $\mathbf{B}$ often takes place in the presence of a strong large-scale magnetic field $\mathbf{B}^o$. It is, for example, what happens in the solar corona where, in spite of the typical value of the sun's magnetic field ($\approx$1 G), fields as intense as $\approx$500 G can be observed in solar flares. These highly energetic and large-scale events coexist with small-scale turbulent activity, finally responsible for the dissipation of magnetic field energy. Modelling the way through which energy is stored and then dissipated is, consequently, not an easy task.

In Ref. [1], the following description is proposed: a large-scale axial, e.g., directed parallel to some vector $\hat{\mathbf{z}}$, magnetic field $\mathbf{B}^o$ is assumed to dominate the dynamics in the $\hat{\mathbf{z}}$ direction, while the activity in the transverse plane can be satisfactorily described as quasibidimensional. This picture allows reliable numerical simulations in two dimensions, from which it appears clear that the magnetic field tends to organize in rare large-scale structures separated by narrow current sheets. Deep investigation of small-scale intermittency properties is still not permitted by lack of spatial resolution.

An interesting question raised by this problem, besides structure formation, is related to the role played by large-scale anisotropy on the small-scale statistics. Indeed, this is quite a typical situation in turbulence, where almost every large-scale forcing is not isotropic. Here, instead of taking the restoration of local small-scale isotropy for granted, as in the Kolmogorov theory of turbulence [2–4], we analyze in detail the effects of anisotropic large-scale contributions on the small-scale magnetic fluctuations.

A wide interest has been recently devoted to this issue [5–17]. From the viewpoints of theoretical and numerical analysis, focusing on a small number of indicators some arguments are given in favor of the small-scale isotropy restoration in the Navier-Stokes (NS) turbulence [12,14]. On the other hand, investigating a larger class of anisotropic indicators, footprints of small-scale anisotropy become manifest [9]. For a passively advected scalar, experiments [5,6] and analytical results [10,11] show that the skewness factor remains $O(1)$ deep in the inertial range. The scenario thus appears extremely faceted and needs further investigations.

Recently, clear evidence of persistent small-scale anisotropy has been found in Ref. [17], where the statistical properties of a scalar field advected by the nonintermittent NS flow generated in a two-dimensional inverse cascade regime are investigated.

Two main goals motivate this paper. On one hand, we give details of the results presented in the Rapid Communication [16], where the effects of anisotropy on scaling exponents of the two-point magnetic field correlations have been addressed in the framework of the kinematic magnetohydrodynamics (MHD) problem. Nonperturbative expressions for the scaling exponents were derived and their universality proved. Specifically, there arises a picture of a nontrivial statistical behavior, where anisotropic fluctuations are organized in a hierarchical order according to their degree of anisotropy. Contributions belonging to shells of higher anisotropic index decay faster, and the isotropic contribution finally dominates.

However, the dominance of the isotropic contribution in the scaling exponents does not imply that large-scale anisotropy is irrelevant for the small-scale magnetic statistics. A deep investigation focused on a larger number of statistical indices (that is focused on the proper anisotropy indicators) has to be performed in order to highlight the way (if any) through which large-scale anisotropy manifests itself at small scales. This is the second aim of the present paper. Specifically, in addition to the nonperturbative results for the two-point correlations, we present new results dealing with higher-order magnetic correlation functions. Being more specific, we exploit the field theoretic renormalization group (RG) to obtain the anomalous exponents for higher-order magnetic correlation functions at the first order in ξ, the exponent entering into the velocity covariance. In particular, we evaluate the odd-order correlation function exponents, from which dimensionless ratios like skewness and hyperskewness are calculated. As a result, in three dimensions, the former behaves at the dissipative scale as $\mathrm{Pe}^{-1/10}$ while the latter as $\mathrm{Pe}^{11/10}$, Pe being the Péclet number (i.e., the equivalent of the Reynolds number for the NS turbulence). Notice the opposite signs appearing in the scaling exponents. They are the signature of persistent small-scale activities. Indeed, the first index is weakly scale dependent while the second is even divergent at small scales (i.e., $\mathrm{Pe}\to\infty$). Let us remark that to restore isotropy at small scales all such indices should decay to zero as Pe grows.

The same general picture is found numerically in Ref. [17] in the framework of the passive scalar advection by NS flows. In addition, our results are in qualitative agreement both with the first-order analytic expressions for the anomalous exponents obtained in [15] for the passive scalar advected by a synthetic velocity field, and with the results of Ref. [8] where the probability density functions of both a scalar field and its gradient are investigated for the class of synthetic fields in the Batchelor regime.

The paper is organized as follows. In Sec. II, we give the detailed definition of the kinematic MHD Kasantzev-Kraichnan model, which describes the passive advection of the magnetic field by the Gaussian, self-similar velocity field δ-correlated in time. In Sec. III, the field theoretic formulation of the model is presented. It allows for the derivation of the closed exact equations for the response function and equal-time pair correlation function of the magnetic field. From the homogeneous solutions (zero modes) of the pair correlation equation, scaling exponents of the pair correlation function are determined. In Sec. IV, these exponents are found nonperturbatively, for any ξ and space dimensionality d. In Sec. V, we discuss the UV renormalization of the model and derive the corresponding β functions and RG equations. The latter possess an infrared (IR) stable fixed point, which establishes the existence of anomalous scaling for all the higher-order correlation functions. The inertial-range behavior of these functions is determined by the scaling dimensions of certain tensor composite operators; they are calculated in Sec. VI to the first order in ξ (one-loop approximation). In Sec. VII, we employ the operator product expansion to give explicit inertial range expressions for various higher-order correlation functions. The results obtained are reviewed in Sec. VIII, where a brief comparison with the passive scalar problem is also given.

II. DEFINITION OF THE KINEMATIC MHD KASANTZEV-KRAICHNAN MODEL

In the presence of a mean component $\mathbf{B}^o$ (actually supposed to be varying on a very large scale $\sim L$, the largest one in our problem) the kinematic MHD equations describing the evolution of the fluctuating part $\mathbf{B}\equiv\mathbf{B}(x)$ of the magnetic field are [18]

$$\partial_t B_\alpha+\mathbf{v}\cdot\boldsymbol{\partial}B_\alpha=\mathbf{B}\cdot\boldsymbol{\partial}v_\alpha+\mathbf{B}^o\cdot\boldsymbol{\partial}v_\alpha+\kappa_0\partial^2 B_\alpha,\quad \alpha=1,\ldots,d. \tag{2.1}$$

Here and below $x\equiv\{t,\mathbf{x}\}$, $\boldsymbol{\partial}\equiv\{\partial_\alpha=\partial/\partial x_\alpha\}$, $\partial^2\equiv\partial_\alpha\partial_\alpha$ is the Laplace operator, d is the dimensionality of the $\mathbf{x}$ space, and $\mathbf{v}=\mathbf{v}(x)$ is the velocity field. Both $\mathbf{v}$ and $\mathbf{B}$ are divergence-free (solenoidal) vector fields: $\partial_\alpha v_\alpha=\partial_\alpha B_\alpha=0$. Equation (2.1) follows from the simplest form of Ohm's law for conductive moving medium, $\mathbf{j}=\sigma(\mathbf{E}+\mathbf{v}\times\mathbf{B}/c)$, and the Maxwell equations neglecting the displacement current: $\partial_t\mathbf{B}/c+\boldsymbol{\partial}\times\mathbf{E}=\mathbf{0}$, $\boldsymbol{\partial}\times\mathbf{B}=4\pi\mathbf{j}/c$ and $\boldsymbol{\partial}\cdot\mathbf{B}=\mathbf{0}$. Here c is the speed of light, $\mathbf{j}$ is the density of the electric current, σ is the conductivity, and $\kappa_0\equiv c^2/4\pi\sigma$ is the magnetic diffusivity. The term $\mathbf{B}^o\cdot\boldsymbol{\partial}v_\alpha$ in (2.1) effectively plays the same role as an external forcing driving the system and being also a source of anisotropy for the magnetic field statistics.

In the real problem, $\mathbf{v}$ obeys the NS equation with the additional Lorentz force term $\propto(\boldsymbol{\partial}\times\mathbf{B})\times\mathbf{B}$, which describes the effects of the magnetic field on the velocity field. The framework of our analysis is the kinematic MHD problem, where the reaction of the magnetic field $\mathbf{B}$ on the velocity field $\mathbf{v}$ is neglected. We assume that at the initial stages $\mathbf{B}$ is weak and does not affect the motions of the conducting fluid: it becomes then a natural assumption to consider the dynamics linear in the magnetic field strength [18]. It is also noteworthy that in more realistic models of the MHD turbulence the magnetic field indeed behaves as a passive vector in the so-called kinetic fixed point of the RG equations (see Refs. [19,20]).

For general velocity fields the well-known closure problem arises even for the kinematic model. This means that the equations of evolution for the single-time multiple-space moments such as $\langle B_\alpha(t,\mathbf{r}_1)\cdots B_\lambda(t,\mathbf{r}_n)\rangle$ are not closed. The situation changes for random velocity fields δ-correlated in time. The physical choice of a real turbulent flow governed by the NS equation is then replaced by an incompressible, self-similar advecting field, with Gaussian statistics and rapidly changing (δ-correlated) in time. This last property allows us to write closed equations for the moments of the magnetic field $\mathbf{B}$ and to perform analytical (both perturbative and nonperturbative) approaches to the d-dimensional problem. Indeed, in the presence of a random velocity field δ-correlated in time, the solution is a Markov process in the time variable and closed moment equations, sometimes called ''Hopf equations,'' can be obtained in analogy to the passive scalar case [21]. Such models have attracted enormous attention recently (see, e.g., Refs. [11,22–26] and references therein) because of the insight they offer into the origin of intermittency and anomalous scaling in fully developed turbulence. We also note that the isotropic version of the kinematic rapid-change magnetic model dates back to 1967 (see Ref. [27]) and was studied by the authors in Refs. [28–31].

More precisely, we shall consider a simplified model in which $\mathbf{v}(x)$ is a Gaussian random field, homogeneous, isotropic and δ-correlated in time, with zero mean and covariance

$$\langle v_\alpha(x) v_\beta(x')\rangle = \delta(t-t') K_{\alpha\beta}(\mathbf{r}) \tag{2.2a}$$

with

$$K_{\alpha\beta}(\mathbf{r}) = D_0 \int \frac{d\mathbf{k}}{(2\pi)^d} \frac{P_{\alpha\beta}(\mathbf{k})}{k^{d+\xi}} \exp[i\mathbf{k}\cdot\mathbf{r}], \quad \mathbf{r}\equiv\mathbf{x}-\mathbf{x}', \tag{2.2b}$$

where $P_{\alpha\beta}(\mathbf{k}) = \delta_{\alpha\beta} - k_\alpha k_\beta/k^2$ is the transverse projector, $\mathbf{k}$ is the momentum, $k\equiv|\mathbf{k}|$, $D_0>0$ is an amplitude factor, and $0<\xi<2$ is a free parameter. The IR regularization is provided by the cutoff in the integral (2.2) from below at $k \simeq m$, where $m\equiv 1/L$ is the reciprocal of the integral turbulence scale; the precise form of the cutoff is not essential. For $0<\xi<2$, the difference

$$S_{\alpha\beta}(\mathbf{r}) \equiv K_{\alpha\beta}(\mathbf{0}) - K_{\alpha\beta}(\mathbf{r}) \tag{2.3}$$

has a finite limit for $m\to 0$:

$$S_{\alpha\beta}(\mathbf{r}) = D r^\xi \left[(d+\xi-1)\,\delta_{\alpha\beta} - \xi \frac{r_\alpha r_\beta}{r^2} \right], \tag{2.4}$$

with

$$D = \frac{-D_0 \Gamma(-\xi/2)}{(4\pi)^{d/2} 2^\xi (d+\xi) \Gamma(d/2+\xi/2)},$$

where $\Gamma(\cdots)$ is the Euler Γ function (note that $D>0$). It follows from Eq. (2.4) that ξ can be viewed as a kind of Hölder exponent, which measures the roughness of the velocity field. In the RG approach, the exponent ξ plays the same role as the parameter $\epsilon=4-d$ does in the RG theory of critical phenomena [32]. The relations

$$g_0 \equiv D_0/\kappa_0 \equiv \Lambda^\xi \tag{2.5}$$

define the coupling constant g_0 (i.e., the expansion parameter in the ordinary perturbation theory) and the characteristic ultraviolet (UV) momentum scale Λ.

III. FIELD THEORETIC FORMULATION OF THE MODEL: DYSON EQUATIONS FOR THE PAIR CORRELATION FUNCTIONS

The stochastic problem (2.1), (2.2) is equivalent to the field theoretic model of the set of three fields $\Phi\equiv\{\mathbf{B}',\mathbf{B},v\}$ with action functional

$$S(\Phi) = \mathbf{B}'[-\partial_t \mathbf{B} - (\mathbf{v}\cdot\boldsymbol{\partial})\mathbf{B} + (\mathbf{B}\cdot\boldsymbol{\partial})\mathbf{v} + (\mathbf{B}^o\cdot\boldsymbol{\partial})\mathbf{v} + \kappa_0 \partial^2 \mathbf{B}] - \mathbf{v}K^{-1}\mathbf{v}/2. \tag{3.1}$$

The first five terms represent the Martin-Siggia-Rose action (see, e.g., Refs. [32–34]) for the stochastic problem (2.1) at fixed $\mathbf{v}$, and the last term represents the Gaussian averaging over $\mathbf{v}$; K^{-1} is the inverse integral operation for (2.2b) and $\mathbf{B}'$ is a solenoidal response vector field. In (3.1) and analogous formulas below, the required integrations over $\{t,\mathbf{x}\}$ and summations over the vector indices are implied, for example,

$$\mathbf{B}'\partial_t \mathbf{B} \equiv \int dt \int d\mathbf{x}\, B'_\alpha(x)\partial_t B_\alpha(x),$$

$$\mathbf{v}K^{-1}\mathbf{v} \equiv \int dt \int d\mathbf{x} \int d\mathbf{x}'\, v_\alpha(t,\mathbf{x}) K^{-1}_{\alpha\beta}(\mathbf{x}-\mathbf{x}') v_\beta(t,\mathbf{x}').$$

The formulation (3.1) means that statistical averages of random quantities in the stochastic problem (2.1), (2.2) coincide with functional averages with the weight $\exp S(\Phi)$. The model (3.1) corresponds to a standard Feynman diagrammatic technique with the triple vertex $\mathbf{B}'[-(\mathbf{v}\cdot\boldsymbol{\partial})\mathbf{B} + (\mathbf{B}\cdot\boldsymbol{\partial})\mathbf{v}] = B'_\alpha B_\beta v_\gamma V_{\alpha\beta\gamma}$ with vertex factor

$$V_{\alpha\beta\gamma}(\mathbf{k},\mathbf{p},\mathbf{q}) = ik_\gamma \delta_{\alpha\beta} - ik_\beta \delta_{\alpha\gamma} = -ip_\gamma \delta_{\alpha\beta} + iq_\beta \delta_{\alpha\gamma}, \tag{3.2}$$

where $\mathbf{k}$, $\mathbf{p}$, and $\mathbf{q}$ are the momenta flowing into the vertex via the fields $\mathbf{B}'$, $\mathbf{B}$, and $\mathbf{v}$ respectively. Strictly speaking, the vertex $V_{\alpha\beta\gamma}$ has to be contracted with three transverse projectors, but we omitted them in order to simplify the notation. In most cases, transversality of $V_{\alpha\beta\gamma}$ with respect to all its indices will be restored automatically owing to the contraction with bare propagators. The latter in the frequency-momentum $(\omega,\mathbf{k})$ representation have the form:

$$\langle B_\alpha(\omega,\mathbf{k}) B'_\beta(-\omega,-\mathbf{k})\rangle_0 = \langle B'_\alpha(\omega,\mathbf{k}) B_\beta(-\omega,-\mathbf{k})\rangle_0^* = \frac{1}{(-i\omega+\kappa_0 k^2)} P_{\alpha\beta}(\mathbf{k}),$$

$$\begin{aligned}\langle B_\alpha(\omega,\mathbf{k}) B_\beta(-\omega,-\mathbf{k})\rangle_0 &= \langle \mathbf{B}^o\cdot\mathbf{k}\rangle^2 \langle B_\alpha(\omega,\mathbf{k}) B'_{\alpha'}(-\omega,-\mathbf{k})\rangle_0 \\ &\times \langle v_{\alpha'}(\omega,\mathbf{k}) v_{\beta'}(-\omega,-\mathbf{k})\rangle_0 \\ &\times \langle B'_{\beta'}(\omega,\mathbf{k}) B_\beta(-\omega,-\mathbf{k})\rangle_0,\end{aligned}$$

$$\begin{aligned}\langle B_\alpha(\omega,\mathbf{k}) v_\beta(-\omega,-\mathbf{k})\rangle_0 &= (\mathbf{B}^o\cdot\mathbf{k}) \langle B_\alpha(\omega,\mathbf{k}) B'_{\alpha'}(-\omega,-\mathbf{k})\rangle_0 \\ &\times \langle v_{\alpha'}(\omega,\mathbf{k}) v_\beta(-\omega,-\mathbf{k})\rangle_0,\end{aligned}$$

$$\langle B'_\alpha(\omega,\mathbf{k}) B'_\beta(-\omega,-\mathbf{k})\rangle_0 = 0, \tag{3.3}$$

and the bare propagator $\langle v_\alpha v_\beta\rangle_0$ is given by Eqs. (2.2).

The magnitude $B^o\equiv|\mathbf{B}^o|$ can be eliminated from the action (3.1) by rescaling of the fields: $\mathbf{B}\to B^o\mathbf{B}$, $\mathbf{B}'\to\mathbf{B}'/B^o$. Therefore, any total or connected Green function of the form $\langle \mathbf{B}(x_1)\cdots\mathbf{B}(x_n)\mathbf{B}'(y_1)\cdots\mathbf{B}'(y_p)\rangle$ contains the factor of $(B^o)^{n-p}$. The parameter B^o appears in the bare propagators (3.3) only in the numerators. It then follows that the Green functions with $n-p<0$ vanish identically. On the contrary, the 1-irreducible function $\langle \mathbf{B}(x_1)\cdots\mathbf{B}(x_n)\mathbf{B}'(y_1)\cdots\mathbf{B}'(y_p)\rangle_{1-ir}$ contains a factor of $(B^o)^{p-n}$ and therefore vanishes for $n-p>0$; this fact will be relevant in the analysis of the renormalizability of the model (see Sec. V).

The δ-correlated in-time character of v permits us to exploit the Gaussian integration by parts (a comprehensive description of this technique can be found, e.g., in Ref. [4]) to obtain closed, exact equations for the equal-time correlation functions of the field $\mathbf{B}$. This strategy has been used in Ref. [16]. Below we give an alternative derivation of the equation for the pair correlation functions based on the field theoretical formulation of the problem (see also Ref. [30] for the scalar case).

The pair correlation functions $\langle\Phi\Phi\rangle$ of the multicomponent field Φ satisfy the standard Dyson equation, which in the component notation reduces to the system of two nontrivial equations for the exact correlation function $C_{\alpha\beta}(\omega,\mathbf{k})=\langle B_\alpha(\omega,\mathbf{k})B_\beta(-\omega,-\mathbf{k})\rangle$ and the exact response function $G_{\alpha\beta}(\omega,\mathbf{k})=\langle B_\alpha(\omega,\mathbf{k})B'_\beta(-\omega,-\mathbf{k})\rangle$. The latter is independent of $\mathbf{B}^o$ (see above) and thus can be written as $G_{\alpha\beta}(\omega,\mathbf{k})=P_{\alpha\beta}(\mathbf{k})G(\omega,k)$ with a certain isotropic scalar function $G(\omega,k)$. In our model these equations, usually referred to as the Dyson-Wyld equations (see, e.g., Ref. [3]), have the form

$$G^{-1}(\omega,k)P_{\alpha\beta}(\mathbf{k})=[-i\omega+\kappa_0k^2]P_{\alpha\beta}(\mathbf{k})-\Sigma^{B'B}_{\alpha\beta}(\omega,\mathbf{k}), \tag{3.4a}$$

$$C_{\alpha\beta}(\omega,\mathbf{k})=|G(\omega,k)|^2[(\mathbf{B}^o\cdot\mathbf{k})^2\langle v_\alpha(\omega,\mathbf{k})v_\beta(-\omega,-\mathbf{k})\rangle_0 +\Sigma^{B'B'}_{\alpha\beta}(\omega,k)], \tag{3.4b}$$

where $\langle B_\alpha B_\beta\rangle_0$ is given in Eq. (3.3), $\Sigma^{B'B}$ and $\Sigma^{B'B'}$ are self-energy operators represented by the corresponding 1-irreducible diagrams; the other functions $\Sigma^{\Phi\Phi}$ vanish identically. It is also convenient to contract Eq. (3.4a) with the projector $P_{\alpha\beta}(\mathbf{k})$ in order to obtain the scalar equation

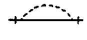
$$G^{-1}(\omega,k)=-i\omega+\kappa_0k^2-\Sigma^{B'B}(\omega,k), \tag{3.5a}$$

where we have written

$$\Sigma^{B'B}(\omega,k)\equiv\Sigma^{B'B}_{\alpha\beta}(\omega,\mathbf{k})P_{\alpha\beta}(\mathbf{k})/(d-1). \tag{3.5b}$$

The feature characteristic of the rapid-change models like (3.1) is that all the skeleton multiloop diagrams entering into the self-energy operators $\Sigma^{B'B}$ and $\Sigma^{B'B'}$ contain effectively closed circuits of retarded propagators $\langle\mathbf{B}\mathbf{B}'\rangle$ and therefore vanish; it is also crucial here that the propagator $\langle\boldsymbol{v}\boldsymbol{v}\rangle$ in Eq. (2.2a) is proportional to the δ function in time. Therefore the self-energy operators in (3.4) are given by the one-loop approximation exactly and have the form

$\Sigma^{B'B}=$  , (3.6a)

$\Sigma^{B'B'}=$  . (3.6b)

The solid lines in the diagrams denote the exact propagators $\langle\mathbf{B}\mathbf{B}'\rangle$ and $\langle\mathbf{B}\mathbf{B}\rangle$; the ends with a slash correspond to the field $\mathbf{B}'$, and the ends without a slash correspond to $\mathbf{B}$; the dashed lines denote the velocity propagator (2.2); the vertices correspond to the factor (3.2). The analytic expressions for the diagrams in Eq. (3.6) have the form

$$\Sigma^{B'B}(\omega,k)=\frac{P_{\alpha\beta}(\mathbf{k})}{(d-1)}\int\frac{d\omega'}{2\pi}\times\int\frac{d\mathbf{q}}{(2\pi)^d}V_{\alpha\alpha_3\alpha_1}(\mathbf{k},\mathbf{p},\mathbf{q})P_{\alpha_3\alpha_4}(\mathbf{p})G(\omega',\mathbf{p})\times\frac{D_0P_{\alpha_1\alpha_2}(\mathbf{q})}{q^{d+\xi}}V_{\alpha_4\beta\alpha_2}(-\mathbf{k},-\mathbf{p},-\mathbf{q}), \tag{3.7a}$$

$$\Sigma^{B'B'}_{\alpha\beta}(\omega,\mathbf{k})=\int\frac{d\omega'}{2\pi}\int\frac{d\mathbf{q}}{(2\pi)^d}V_{\alpha\alpha_3\alpha_1}(\mathbf{k},\mathbf{p},\mathbf{q})C_{\alpha_3\alpha_4}(\omega',\mathbf{p})\times\frac{D_0P_{\alpha_1\alpha_2}(\mathbf{q})}{q^{d+\xi}}V_{\beta\alpha_4\alpha_2}(-\mathbf{k},-\mathbf{p},-\mathbf{q}), \tag{3.7b}$$

where $\mathbf{k}+\mathbf{q}+\mathbf{p}=\mathbf{0}$, the vertex $V_{\alpha\beta\gamma}$ is defined in Eq. (3.2), and the explicit form (2.2) of the velocity covariance is used. We also recall that the integrations over $\mathbf{q}$ should be cut off from below at $q=m$.

The integrations over ω' in the right-hand sides of Eqs. (3.7) give the equal-time response function $G(q)=(1/2\pi)\int d\omega' G(\omega',q)$ and the equal-time pair correlation function $C_{\alpha\beta}(\mathbf{q})=(1/2\pi)\int d\omega' C_{\alpha\beta}(\omega',\mathbf{q})$; note that both the self-energy operators are in fact independent of ω. The only contribution to $G(q)$ comes from the bare propagator (3.3), which in the t representation is discontinuous at coincident times. Since the correlation function (2.2a), which enters into the one-loop diagram for $\Sigma^{B'B}$, is symmetric in t and t', the response function must be defined at $t=t'$ by half the sum of the limits. This is equivalent to the convention

$$G(q)=(1/2\pi)\int d\omega'(-i\omega'+\kappa_0q^2)^{-1}=1/2$$

and gives

$$\Sigma^{B'B}(\omega,k)=\frac{P_{\alpha\beta}(\mathbf{k})}{2(d-1)}\int\frac{d\mathbf{q}}{(2\pi)^d}V_{\alpha\alpha_3\alpha_1}(\mathbf{k},\mathbf{p},\mathbf{q})\times P_{\alpha_3\alpha_4}(\mathbf{p})\frac{D_0P_{\alpha_1\alpha_2}(\mathbf{q})}{q^{d+\xi}}\times V_{\alpha_4\beta\alpha_2}(-\mathbf{k},-\mathbf{p},-\mathbf{q}). \tag{3.8}$$

Substituting Eq. (3.2) into Eq. (3.8) after lengthy but straightforward calculations gives

$$\Sigma^{B'B}(\omega,k)=(-1/2)k_\alpha k_\beta D_0\int\frac{d\mathbf{q}}{(2\pi)^d}\frac{P_{\alpha\beta}(\mathbf{q})}{q^{d+\xi}}. \tag{3.9}$$

The integration over $\mathbf{q}$ in Eq. (3.9) is performed explicitly using the relation

$$\int d\mathbf{q}f(q)\frac{q_\alpha q_\beta}{q^2}=\frac{\delta_{\alpha\beta}}{d}\int d\mathbf{q}f(q) \tag{3.10}$$

and gives

$$\Sigma^{B'B}(\omega,k)=-k^2\frac{D_0(d-1)}{2d}J(m), \tag{3.11a}$$

where we have written

$$J(m)\equiv\int\frac{d\mathbf{q}}{(2\pi)^d}\frac{1}{q^{d+\xi}}=C_d m^{-\xi}/\xi. \tag{3.11b}$$

Here and below $C_d\equiv S_d/(2\pi)^d$ and $S_d\equiv 2\pi^{d/2}/\Gamma(d/2)$ is the surface area of the unit sphere in d-dimensional space; the parameter m has arisen from the lower limit in the integral over $\mathbf{q}$.

Equations (3.5), (3.11) give an explicit exact expression for the response function in our model; it will be used in Sec. V for the exact calculation of the RG functions. Like in the scalar case, the exact response function differs from its bare analog (3.3) simply by the substitution $\kappa_0\to\kappa_0+D_0(d-1)J(m)/2d$. Below we use the intermediate expression (3.9). The integration of Eq. (3.4b) over the frequency ω gives a closed equation for the equal-time correlation function; it is important here that the ω dependence of the right-hand side is contained only in the prefactor $|G(\omega,k)|^2$. Using Eq. (3.9) the equation for $C_{\alpha\beta}(\mathbf{k})$ can be written in the form

$$\begin{aligned}2(\kappa_0 k^2+\Sigma^{B'B})C_{\alpha\beta}(\mathbf{k})=&(\mathbf{B}^o\cdot\mathbf{k})^2\langle v_\alpha(\omega,\mathbf{k})v_\beta(-\omega,-\mathbf{k})\rangle_0\\&+\int\frac{d\mathbf{q}}{(2\pi)^d}V_{\alpha\alpha_3\alpha_1}(\mathbf{k},\mathbf{p},\mathbf{q})\\&\times C_{\alpha_3\alpha_4}(\mathbf{p})\frac{D_0P_{\alpha_1\alpha_2}(\mathbf{q})}{q^{d+\xi}}\\&\times V_{\beta\alpha_4\alpha_2}(-\mathbf{k},-\mathbf{p},-\mathbf{q}),\end{aligned} \tag{3.12}$$

and using Eqs. (3.2) and (3.9) it can be rewritten as

$$\begin{aligned}2\kappa_0k^2C_{\alpha\beta}(\mathbf{k})=&(\mathbf{B}^o\cdot\mathbf{k})^2\langle v_\alpha(\omega,\mathbf{k})v_\beta(-\omega,-\mathbf{k})\rangle_0\\&+\int\frac{d\mathbf{q}}{(2\pi)^d}\frac{D_0}{q^{d+\xi}}\{q_{\alpha_1}q_{\alpha_2}C_{\alpha_1\alpha_2}(\mathbf{p})P_{\alpha\beta}(\mathbf{q})\\&-p_{\alpha_1}q_{\alpha_2}C_{\alpha\alpha_2}(\mathbf{p})P_{\alpha_1\beta}(\mathbf{q})\\&-p_{\alpha_2}q_{\alpha_1}C_{\alpha_1\beta}(\mathbf{p})P_{\alpha\alpha_2}(\mathbf{q})\}\\&+\int\frac{d\mathbf{q}}{(2\pi)^d}\frac{D_0P_{\alpha_1\alpha_2}(\mathbf{q})}{q^{d+\xi}}\{p_{\alpha_1}p_{\alpha_2}C_{\alpha\beta}(\mathbf{p})\\&-k_{\alpha_1}k_{\alpha_2}C_{\alpha\beta}(\mathbf{k})\}.\end{aligned} \tag{3.13}$$

For $0<\xi<2$, Eq. (3.13) allows for the limit $m\to 0$: the first three integrals in its right-hand side are separately finite for $m=0$; the last integral is finite owing to the subtraction, which has come from the contribution with $\Sigma^{B'B}$ in the left-hand side of Eq. (3.12). Indeed, the possible IR divergence of this integral at $\mathbf{q}=0$ is suppressed by the vanishing of the expression in the curly brackets. In what follows we set $m=0$.

Equation (3.13) can also be rewritten as a partial differential equation for the pair correlation function in the coordinate representation, $C_{\alpha\beta}(\mathbf{r})\equiv\langle B_\alpha(t,\mathbf{x})B_\beta(t,\mathbf{x}+\mathbf{r})\rangle$ [we use the same notation $C_{\alpha\beta}$ for the coordinate function and its Fourier transform]. Noting that the integrals in Eq. (3.13) involve convolutions of the functions $C_{\alpha\beta}(\mathbf{k})$ and $D_0P_{\alpha\beta}(\mathbf{k})/k^{d+\xi}$, the Fourier transform of the spatial part (2.2b) of the velocity correlation function (2.2), and replacing the momenta by the corresponding derivatives, $ip_\alpha\to\partial_\alpha$ and so on, we obtain

$$\begin{aligned}2\kappa_0\partial^2C_{\alpha\beta}=&-(\partial_{\alpha_1}\partial_{\alpha_2}S_{\alpha\beta})(B^o_{\alpha_1}B^o_{\alpha_2}+C_{\alpha_1\alpha_2})\\&+(\partial_{\alpha_2}S_{\alpha_1\beta})\partial_{\alpha_1}C_{\alpha\alpha_2}+(\partial_{\alpha_1}S_{\alpha\alpha_2})(\partial_{\alpha_2}C_{\alpha_1\beta})\\&-S_{\alpha_1\alpha_2}\partial_{\alpha_1}\partial_{\alpha_2}C_{\alpha\beta}.\end{aligned} \tag{3.14}$$

Note that the correlation function (2.2) enters into Eq. (3.14) only through the function $S_{\alpha_1\alpha_2}$ from Eq. (2.4), or, in other words, through the difference (2.3), which has a finite limit at $m=0$. The m dependent constant part of (2.2b) vanishes under the differentiation in the first four terms in the right-hand side of Eq. (3.14), and in the last term it is subtracted explicitly, owing to the subtraction in Eq. (3.13). Equation (3.14) should be augmented by the solenoidality condition:

$$\partial_\alpha C_{\alpha\beta}=0. \tag{3.15}$$

For the nonstationary state, the function $C_{\alpha\beta}(t,\mathbf{r})\equiv\langle B_\alpha(t,\mathbf{x})B_\beta(t,\mathbf{x}+\mathbf{r})\rangle$ depends explicitly on t, and the term $\partial_t C_{\alpha\beta}$ appears on the right-hand side of Eq. (3.14), see, e.g., Ref. [16].

IV. NONPERTURBATIVE RESULTS FOR THE SCALING EXPONENTS OF THE TWO-POINT MAGNETIC CORRELATION FUNCTION

In this section we focus our attention on the inertial-range behavior of the second-order equal-time correlation function $C_{\alpha\beta}(t,\mathbf{r})\equiv\langle B_\alpha(t,\mathbf{r})B_\beta(t,\mathbf{0})\rangle$ in the statistically steady state. As shown in Ref. [16], a steady state is present when $\xi<1$, $\xi=1$ being the threshold of instability. As such threshold coincides with that of the isotropic problem [28], it follows that dynamo effect is thus not switched on by anisotropic contributions.

In the isotropic case, the analytic expression for the scaling exponent of $C_{\alpha\beta}$ has been obtained in Ref. [28]. It was also shown by the author of [28] that the anomalous exponent is universal, and the anomaly is associated with zero-mode solutions of the equations satisfied by $C_{\alpha\beta}$. Higher-order correlation function exponents have been calculated to $O(\xi)$ in Ref. [30] by exploiting the RG.

With respect to Ref. [28], the main technical difference is that, in order to extract the anisotropic contributions to the isotropic scaling, the angular structure of zero modes has now to be explicitly taken into account. To start our analysis, let us consider the closed Eq. (3.14) for $C_{\alpha\beta}$. For what follows, it is worth emphasizing two properties of $C_{\alpha\beta}$.

(i) Because of homogeneity, $C_{\alpha\beta}$ is left invariant under the following set of transformations:

$$r\mapsto -r \quad\text{and}\quad \alpha\leftrightarrow\beta; \tag{4.1}$$

(ii) $C_{\alpha\beta}(\mathbf{r})=C_{\alpha\beta}(-\mathbf{r})$, as it follows from (3.14) after the substitution $\mathbf{r}\mapsto-\mathbf{r}$.

In the presence of anisotropy, the most general expression for the two-point magnetic correlations, $C_{\alpha\beta}(\mathbf{r})$, in the stationary state involves five (two in the isotropic case) functions depending on both $r\equiv|\mathbf{x}-\mathbf{x}'|$ and $z\equiv\cos\theta=\hat{\mathbf{B}}^o\cdot\mathbf{r}/r$, where $\hat{\mathbf{B}}^o$ is the unit vector corresponding to the direction selected by the mean magnetic field. Note that the space is anisotropic but still homogeneous, so that there is no explicit dependence on the points $\mathbf{x},\mathbf{x}'$, but only on their difference. Namely,

$$C_{\alpha\beta}(\mathbf{r})=\mathcal{F}_1(r,z)\,\frac{r_\alpha r_\beta}{r^2}+\mathcal{F}_2(r,z)\,\delta_{\alpha\beta}+\mathcal{F}_3(r,z)\,\frac{\hat{B}^o_\alpha r_\beta}{r}+\mathcal{F}_4(r,z)\,\frac{\hat{B}^o_\beta r_\alpha}{r}+\mathcal{F}_5(r,z)\hat{B}^o_\alpha\hat{B}^o_\beta. \quad (4.2)$$

From the properties (i) and (ii) of $C_{\alpha\beta}(\mathbf{r})$ one immediately obtains the following relations for the $\mathcal{F}$'s:

$$\mathcal{F}_i(r,z)=\mathcal{F}_i(r,-z)\quad i=1,2,5, \quad (4.3)$$

$$\mathcal{F}_3(r,z)=-\mathcal{F}_3(r,-z), \quad (4.4)$$

$$\mathcal{F}_3(r,z)=\mathcal{F}_4(r,z). \quad (4.5)$$

Substituting the expression (4.2) into (3.14) and using the chain rules, we obtain, after lengthy but straightforward algebra, the following four equations (corresponding to the projections over $r_\alpha r_\beta/r^2$, $\delta_{\alpha\beta}$, $\hat{B}^o_\alpha r_\beta/r$, and $\hat{B}^o_\alpha\hat{B}^o_\beta$):

$$[a_1r^2\partial_r^2+b_1r\partial_r+c_1(1-z^3)\partial_z^2+d_1z\partial_z+e_1]\mathcal{F}_1+[f_1r\partial_r+g_1z\partial_z+j_1]\mathcal{F}_2+[k_1zr\partial_r+l_1z^2\partial_z+m_1z+n_1\partial_z]\mathcal{F}_3+[o_1+p_1z^2]\mathcal{F}_5=(q_1+r_1z^2)B^{o2}, \quad (4.6)$$

$$a_2\mathcal{F}_1+[b_2r^2\partial_r^2+c_2r\partial_r+d_2(1-z^2)\partial_z^2+e_2z\partial_z+f_2]\mathcal{F}_2+g_2z\mathcal{F}_3+[k_2+l_2z^2]\mathcal{F}_5=(m_2+n_2z^2)B^{o2}, \quad (4.7)$$

$$a_3\partial_z\mathcal{F}_1+b_3\partial_z\mathcal{F}_2+[c_3r^2\partial_r^2+d_3r\partial_r+e_3(1-z^2)\partial_z^2+f_3z\partial_z+g_3]\mathcal{F}_3+[j_3zr\partial_r+(k_3+l_3z^2)\partial_z+m_3z]\mathcal{F}_5=n_3B^{o2}z, \quad (4.8)$$

$$a_4\partial_z\mathcal{F}_3+[b_4r^2\partial_r^2+c_4r\partial_r+d_4(1-z^2)\partial_z^2+e_4z\partial_z+f_4]\mathcal{F}_5=g_4B^{o2}, \quad (4.9)$$

where the coefficients $a_i,b_i,\ldots,r_i$ are functions of ξ and d and are reported in Appendix A. Without loss of generality, we have fixed $D=1$ in (2.4), and we have neglected all terms involving the magnetic diffusivity κ_0, our attention being indeed focused in the inertial range of scales, i.e., $\eta\ll r\ll L$, where $\eta=\kappa_0^{1/\xi}\propto\Lambda^{-1}$ is the dissipative scale for the problem.

With the substitution of the expression (4.2), the solenoidal condition (3.15) splits into the following couple of equations:

$$[r\partial_r+(d-1)]\mathcal{F}_1+[r\partial_r-z\partial_z]\mathcal{F}_2+[zr\partial_r+\partial_z-z^2\partial_z-z]\mathcal{F}_3=0, \quad (4.10)$$

$$\partial_z\mathcal{F}_2+[r\partial_r+d]\mathcal{F}_3+[zr\partial_r+(1-z^2)\partial_z]\mathcal{F}_5=0, \quad (4.11)$$

associated to the projections over r_β/r and $\hat{B}^o_\beta$, respectively.

From the relation (4.5), Eqs. (4.10) and (4.11), it then follows that only two functions of the $\mathcal{F}'s$ in (4.2) are independent. A possible way to isolate contributions of the anisotropic components from the isotropic scaling is to use the decomposition of $\mathcal{F}'s$ on the Legendre polynomial basis. This is the subject of the next subsection.

A. Decomposition in Legendre polynomials

In terms of the Legendre polynomials, functions $\mathcal{F}_i(r,z)$ can be decomposed in the form:

$$\mathcal{F}_i(r,z)=\sum_{j=0}^{\infty}f_j^{(i)}(r)P_j(z)\quad i=1,2,5\quad (j\ \text{even}), \quad (4.12)$$

$$\mathcal{F}_3(r,z)=\sum_{j=0}^{\infty}f_j^{(3)}(r)P_j(z)\quad (j\ \text{odd}), \quad (4.13)$$

where the separation of even and odd orders in (4.12) and (4.13) arises as a consequence of the symmetries expressed by the relations (4.3) and (4.4), respectively.

Simple considerations related to the ''uniaxial'' character of the forcing term with $\mathbf{B}^o$ in the basic equation (2.1), and the linearity of the latter in $\mathbf{B}^o$ and $\mathbf{B}$ suggest that the index j in the above decompositions should be restricted to $j\leqslant 2$. The rigorous assessment of this point will be given in Sec. IV B. On the other hand, contributions associated to $j>2$ can be easily ''activated'' either when a fully anisotropic forcing (i.e., projecting onto all Legendre polynomials) is added on the right-hand side of (2.1), or in the framework of finite-size systems led by anisotropic boundary conditions. Moreover, as we shall see in Sec. V, scaling exponents associated to $j>2$ contribute to the inertial-range scaling of higher-order correlation functions involving the product $B_\alpha B_\beta$ at a single spacetime point. The latter property holds also without the invocation of a fully anisotropic forcing on the left-hand side of Eq. (2.1). From all these considerations we shall exploit the general decompositions (4.12) and (4.13) involving all j's.

In order to obtain equations for $f_j^{(i)}(r)$, we have to insert Eqs. (4.12) and (4.13) into Eqs. (4.6)–(4.11). Furthermore, quantities like $z^p\partial_z^q\mathcal{F}$ ($p=0,1,2$, and $q=0,1$) and $(1-z^2)\partial_z^2\mathcal{F}$ have to be expressed in terms of Legendre polynomials. This can be done exploiting well-known relations involving the Legendre polynomials (see, e.g., Ref. [35]): the resulting expressions for $z^p\partial_z^q\mathcal{F}$ and $(1-z^2)\partial_z^2\mathcal{F}$ are reported in Appendix B. For the sake of brevity, we report hereafter only the projection of Eq. (4.6), the structure of the others being indeed similar (the full set of equations is however reported in Appendix C):

$$a_1 r^2 f_j''^{(1)} + b_1 r f_j'^{(1)} + c_1\left[j(1-j) f_j^{(1)} + 2(2j+1)\sum_{q=1}^{\infty} f_{2q+j}^{(1)}\right] + d_1\left[j f_j^{(1)} + (2j+1)\sum_{q=1}^{\infty} f_{2q+j}^{(1)}\right] + e_1 f_j^{(1)} + f_1 r f_j'^{(2)}$$
$$+ g_1\left[j f_j^{(2)} + (2j+1)\sum_{q=1}^{\infty} f_{2q+j}^{(2)}\right] + j_1 f_j^{(2)} + k_1 r\left[\frac{j}{2j-1} f_{j-1}'^{(3)} + \frac{j+1}{2j+3} f_{j+1}'^{(3)}\right] + n_1(2j+1)\sum_{q=0}^{\infty} f_{2q+j+1}^{(3)}$$
$$+ l_1\left[\frac{j(j-1)}{2j-1} f_{j-1}^{(3)} - \frac{(j+1)(j+2)}{2j+3} f_{j+1}^{(3)} + (2j+1)\sum_{q=0}^{\infty} f_{2q+j+1}^{(3)}\right] + m_1\left[\frac{j}{2j-1} f_{j-1}^{(3)} + \frac{j+1}{2j+3} f_{j+1}^{(3)}\right] + o_1 f_j^{(5)}$$
$$+ p_1\left[\frac{j(j-1)}{(2j-3)(2j-3)} f_{j-2}^{(5)} + \left(\frac{(j+1)^2}{(2j+3)(2j+1)} + \frac{j^2}{(2j-1)(2j+1)}\right) f_j^{(5)} + \frac{(j+2)(j+1)}{(2j+3)(2j+5)} f_{j+2}^{(5)}\right]$$
$$= B^{o2}\left[q_1 + r_1\left(\frac{2}{3}\delta_{j,2} + \frac{1}{3}\delta_{j,o}\right)\right]. \tag{4.14}$$

From the above equation we can see that terms like $z^p \partial_z^q \mathcal{F}$ are responsible for the coupling between an arbitrary anisotropic contribution of order j and all larger orders. The full set of equations (C1)–(C6) is thus not closed and there are no chances to solve them analytically. Simple physical argumentations actually permit to overcome the closure problem. Indeed, in the presence of a cascadelike mechanism of energy transfer towards small scales, anisotropy present at the integral scale should rapidly decay during the multiple-step transfer, and an almost isotropic inertial range scaling should be restored. Mathematically, this means that $f_j^{(i)}$'s should be rapidly decreasing functions of the degree of anisotropy j, i.e.,

$$f_1^{(i)} \ll f_2^{(i)} \ll \cdots \tag{4.15}$$

and similarly for their derivatives. We shall control the validity of this physical assumption in a self-consistent way, at the end of our calculation.[1]

The hierarchy (4.15) is exploited here by retaining, for each i appearing in the functions $f_j^{(i)}$'s, the lowest value of the index j. When doing this, the simplifications on Eqs. (C1)–(C6) are enormous and the resulting set of equations reads:

$$a_1 r^2 f_j''^{(1)} + b_1 r f_j'^{(1)} + [c_1 j(1-j) + d_1 j + e_1] f_j^{(1)} + f_1 r f_j'^{(2)}$$
$$+ [g_1 j + j_1] f_j^{(2)} + k_1 r \frac{j}{2j-1} f_{j-1}'^{(3)}$$
$$+ \left[l_1 \frac{j(j-1)}{2j-1} + m_1 \frac{j}{2j-1}\right] f_{j-1}^{(3)}$$
$$+ p_1 \frac{j(j-1)}{(2j-1)(2j-3)} f_{j-2}^{(5)}$$
$$= B^{o2}\left[q_1 + r_1\left(\frac{2}{3}\delta_{j,2} + \frac{1}{3}\delta_{j,o}\right)\right], \tag{4.16}$$

$$a_2 f_j^{(1)} + b_2 r^2 f_j''^{(2)} + c_2 r f_j'^{(2)} + [d_2 j(1-j) + e_2 j + f_2] f_j^{(2)}$$
$$+ g_2 \frac{j}{2j-1} f_{j-1}^{(3)} + l_2 \frac{j(j-1)}{(2j-1)(2j-3)} f_{j-2}^{(5)}$$
$$= B^{o2}\left[m_2 + n_1\left(\frac{2}{3}\delta_{j,2} + \frac{1}{3}\delta_{j,o}\right)\right], \tag{4.17}$$

$$a_3(2j+1) f_{j+1}^{(1)} + b_3(2j+1) f_{j+1}^{(2)} + c_3 r^2 f_j''^{(3)} + d_3 r f_j'^{(3)}$$
$$+ [e_3 j(1-j) + f_3 j + g_3] f_j^{(3)} + j_3 r \frac{j}{2j-1} f_{j-1}'^{(5)}$$
$$+ \left[l_3 \frac{j(j-1)}{2j-1} + m_3 \frac{j}{2j-1}\right] f_{j-1}^{(5)} = n_3 B^{o2}\delta_{j,2}, \tag{4.18}$$

$$a_4(2j+1) f_{j+1}^{(3)} + b_4 r^2 f_j''^{(5)} + c_4 r f_j'^{(5)}$$
$$+ [d_4 j(1-j) + e_4 j + f_4] f_j^{(5)}$$
$$= g_4 B^{o2}\delta_{j,0}, \tag{4.19}$$

$$r f_j'^{(1)} + (d-1) f_j^{(1)} + r f_J'^{(2)} - j f_j^{(2)} + r \frac{j}{2j-1} f_{j-1}'^{(3)}$$
$$- \frac{j^2}{2j-1} f_{j-1}^{(3)} = 0, \tag{4.20}$$

$$(2j+1) f_{j+1}^{(2)} + r f_j'^{(3)} + d f_j^{(3)} + r \frac{j}{2j-1} f_{j-1}'^{(5)} - \frac{j(j-1)}{2j-1} f_{j-1}^{(5)}$$
$$= 0. \tag{4.21}$$

Some remarks are noteworthy. Focusing on the isotropic contribution, $j=0$, we notice that the first two equations involve solely the functions $f_0^{(1)}$ and $f_0^{(2)}$ (and their derivatives). With the solenoidal condition (4.20), it is easy to

[1]The physical assumption (4.15) is unnecessary when the decomposition in the irreducible representations of the SO(d) symmetry group is exploited (see Ref. [36] for the case $d=3$). This leads exactly to the same results obtained earlier in Ref. [16], where the hierarchy (4.15) was assumed. Notice that the additional exponents (subsets II and III) reported in [36] are related to the pseudotensorial structures and that in our model they do not contribute to the inertial-range behavior of the pair correlator.

check that they coincide with the equation reported in Ref. [28] for the isotropic problem. Moreover, for $j\geqslant 2$, Eq. (4.16) suggests taking the function $\mathbf{f}_j \equiv (f_j^{(1)}, f_j^{(2)}, f_{j-1}^{(3)}, f_{j-2}^{(5)})$ as an unknown field. It is immediately verified that $\mathbf{f}_j$ appears also in the other equations when the index $j(j\geqslant 2)$ is renamed $(j-1)$ in Eqs. (4.18) and (4.21) and $(j-2)$ in Eq. (4.19). When doing this, a linear partial differential equation (PDE) system of the type $\mathcal{L}_j\mathbf{f}_j = \mathbf{g}_j$ (hereafter, repeated indices are not summed) is obtained, $\mathbf{g}_j$ involving all terms related to the mean field B^o, and $\mathcal{L}_j$ is restricted, for instance, to the first four equations.

The analytical treatment of the resulting equation system still remains a very hard task for general values of the space separation r. The situation changes when one focuses on the inertial range of scales (i.e., for $\eta \ll r \ll L$). In the latter case scaling behaviors are expected and we shall have

$$f_j^{(i)}(r) \propto r^{\zeta_j^{(i)}} \quad \text{with} \quad \zeta_0^{(i)} < \zeta_1^{(i)} < \cdots, \tag{4.22}$$

where the hierarchy on the exponents $\zeta_j^{(i)}$ immediately follows from (4.15).

The structure of the above equations fixes the relation between the scaling exponents relative to different f's. Indeed, when searching for power law solutions $f_j^{(i)}(r) \propto r^{\zeta_j^{(i)}}$, in order to obtain balanced equations the ''oblique'' relations must hold:

$$\zeta_j \equiv \zeta_j^{(1)} = \zeta_j^{(2)} = \zeta_{j-1}^{(3)} = \zeta_{j-2}^{(5)}. \tag{4.23}$$

We are now ready to show that nontrivial scaling behaviors for $\mathbf{f}_j$ take place due to zero modes, i.e., solutions of the homogeneous problem $\mathcal{L}_j\mathbf{f}_j = \mathbf{0}$. To that purpose, we exploit (4.23) and define coefficients $\mathbf{y}_j$ through the relation $\mathbf{f}_j \equiv \mathbf{y}_j r^{\zeta_j}$. Inserting the latter expression in the PDE system, a 4×4 algebraic linear system for $\mathbf{y}_j$ is obtained. The emergence of zero modes is thus reduced to impose the existence of nontrivial solutions of a 4×4 homogeneous linear system, that means here the resolution of an algebraic equation of eighth degree arising from the condition that the determinant of the system coefficients is zero. The calculation, lengthy but straightforward, leads to four sets of zero modes [actually eight sets, but it turns out that the associated coefficients, $\mathbf{y}_j$, of four of them do not satisfy the solenoidal condition (4.20)–(4.21)] the expressions and the admissibility of which are given and discussed in the next section.

B. Zero-mode solutions and their admissibility

Let us start with the case $j=0$, corresponding to the isotropic contribution. As already observed, Eqs. (4.16) and (4.17) are decoupled from the others, and the problem can be solved directly for $f_0^{(1)}$ and $f_0^{(2)}$, which must satisfy also the solenoidal condition (4.20). The imposition of the existence of nontrivial solutions (for $j=0$ we have a homogeneous 2×2 algebraic linear system for $\mathbf{y}_j$) and the solenoidal constraint (4.20) lead to the following solutions:

$$\zeta_0^{\pm} = \frac{-d^2+d-2\xi \pm \sqrt{12d^2\xi - 8d\xi + 8\xi^2 d - 4d^2\xi^2 - 4d^3\xi + d^2 + d^4 - 2d^3}}{2d-2} \tag{4.24}$$

with their $\xi\to 0$ and $d\to\infty$ limits:

$$\zeta_0^{+} = -\xi + O(\xi^2) = -\xi - \frac{2\xi^2}{d} + O(1/d^2), \tag{4.25}$$

$$\zeta_0^{-} = -d + \xi\frac{d-3}{d-1} + O(\xi^2) = -\frac{1}{d} + \frac{2\xi(\xi-1)}{d} + O(1/d^2). \tag{4.26}$$

For $j\geqslant 2$ the zero-mode exponents are

$$\begin{aligned}\zeta_j^{\pm} = -\frac{1}{2(d-1)}\{&2\xi + d^2 - d - [-2d^3\xi - 2d^2\xi^2 - 6d^3 \\ &+ 4\xi^2 d + 8 + 10d\xi + 20dj - 20d - 8\xi - 8j + 4d^2j^2 \\ &+ 2\xi^2 - 4\xi j^2 + 17d^2 - 8dj^2 + 8\xi j + 4d^3 j + 4d^2 j\xi \\ &+ 4dj^2\xi + 4j^2 - 16d^2 j - 12d\xi j + d^4 \\ &\pm 2\sqrt{K}(d-1)(2-\xi)]^{1/2}\},\end{aligned} \tag{4.27}$$

$$\begin{aligned}\rho_j^{\pm} = -\frac{1}{2(d-1)}\{&2\xi + d^2 - d + [-2d^3\xi - 2d^2\xi^2 - 6d^3 \\ &+ 4\xi^2 d + 8 + 10d\xi + 20dj - 20d - 8\xi - 8j + 4d^2j^2 \\ &+ 2\xi^2 - 4\xi j^2 + 17d^2 - 8dj^2 + 8\xi j + 4d^3 j + 4d^2 j\xi \\ &+ 4dj^2\xi + 4j^2 - 16d^2 j - 12d\xi j + d^4 \\ &\pm 2\sqrt{K}(d-1)(2-\xi)]^{1/2}\},\end{aligned} \tag{4.28}$$

where

$$\begin{aligned}K = (d-1)(&d^3 + 4d^2 j - 5d^2 + 2d^2\xi + \xi^2 d + 4d\xi j - 6d\xi + 8d \\ &- 12dj + 4dj^2 - \xi^2 + 4\xi + 8j - 8\xi j - 4 - 4j^2 + 4\xi j^2),\end{aligned}$$

with their $\xi\to 0$ and $d\to\infty$ limits

$$\zeta_j^{+} = j - \xi\frac{(d-1+j)(d^2+dj-2d+4j-2j^2)}{(d-2+2j)(d+2j)(d-1)} + O(\xi^2) = j - \xi + \frac{2\xi(j-\xi)}{d} + O(1/d^2), \tag{4.29}$$

$$\zeta_j^- = j-2+\xi\frac{-4d^2-17dj+16d+28j-16-14j^2+2d^2j+5dj^2+2j^3}{(d-2+2j)(d+2j-4)(d-1)}+O(\xi^2)$$
$$= j-2+\frac{2\xi(j-2)}{d}+O(1/d^2), \tag{4.30}$$

$$\rho_j^+ = -d-j+\xi\frac{-5d^2-7dj+6d+4j-2j^2+d^3+2d^2j-dj^2-2j^3}{(d-2+2j)(d+2j)(d-1)}+O(\xi^2)$$
$$= -\frac{1}{d}+\xi-j-\frac{2\xi(j+1-\xi)}{d}+O(1/d^2), \tag{4.31}$$

$$\rho_j^- = 2-d-j-\xi\frac{(j-1)(2j^2-4j+5dj-4d+2d^2)}{(d-2+2j)(d+2j-4)(d-1)}+O(\xi^2)$$
$$= -\frac{1}{d}+2-j-\frac{2\xi(j-1)}{d}+O(1/d^2). \tag{4.32}$$

Let us discuss the admissibility of these solutions. With the term admissible we mean a solution $\mathbf{f}_j(r)$ satisfying the appropriate boundary conditions, at both small (UV limit) and large scales (IR limit). Specifically, the following asymptotic behaviors have to be satisfied:

$$\mathbf{f}_j(r) \quad \text{regular for} \quad r\sim\eta\to 0 \tag{4.33}$$
$$\mathbf{f}_j(r)\to 0 \quad \text{for} \quad r\gg L. \tag{4.34}$$

Concerning the limit (4.33), we have to consider solutions corresponding to the diffusive range and to match them with our inertial-range power laws. From Eq. (3.14) we can easily see that the equations holding in the diffusive range are obtained by setting to zero the parameter ξ. The consequence is that our inertial-range zero-mode solutions become solutions in the diffusive range for $\xi=0$. The problem related to the UV boundary condition is thus reduced to search for regular solutions for $\mathbf{f}_j(r)$ in the $\xi\to 0$ limit. This is easy to do and the result is that solely exponents ζ_0^+ and $\zeta_j^\pm$ for $j\geqslant 2$ permit satisfisfaction of the condition of regularity for $\mathbf{f}_j$, the other exponents being indeed $\leqslant 0$ for $\xi=0$. Notice that, the zero-mode exponent ζ_0^+ coincides with the isotropic solution obtained in Ref. [28].

Let us now discuss the IR boundary conditions (4.34). In this case, as pointed out in Ref. [28], a crucial role is played by the external forcing. Indeed, in the presence of forcing, zero modes and the decaying forced solution may be matched at the integral scale L, thus satisfying the IR boundary conditions. The result of this argument [which can be rigorously illustrated solely for $j=0$ where the general solution for $\mathbf{f}_j(r)$ is available] is that zero-mode exponents are not admissible for $j\geqslant 4$. Indeed, as we can see from Eq. (4.16)–(4.19), the forcing term related to B^o projects solely on the shells $j\leqslant 2$.

To summarize, we have one admissible zero mode for $j=0(\zeta_0^+)$ and two admissible zero modes for $j=2(\zeta_2^\pm)$. Our attention being focused on the inertial range of scales (i.e., $r/L\ll 1$), our choice for $j=2$ is for ζ_2^-. We have indeed to take the exponent giving the leading inertial-range contribution.

Finally, we can thus define the final solution ζ_j of our problem as:

$$\zeta_0\equiv\zeta_0^+=\frac{-d^2+d-2\xi+\sqrt{12d^2\xi-8d\xi+8\xi^2d-4d^2\xi^2-4d^3\xi+d^2+d^4-2d^3}}{2d-2}=-\xi+O(\xi^2)=-\xi-\frac{2\xi^2}{d}+O(1/d^2), \tag{4.35}$$

$$\zeta_2\equiv\zeta_2^-=-\frac{1}{2(d-1)}\{d^2-d+2\xi-[8-12d-8\xi+2d\xi+4\xi^2d-2d^2\xi^2-2d^3\xi+8d^2\xi+d^2+2\xi^2+2d^3+d^4$$
$$-2\sqrt{K}(2-\xi)(d-1)]^{1/2}\}$$
$$=\frac{2\xi}{(d-1)(d+2)}+O(\xi^2)$$
$$=\frac{2\xi}{d^2}+O(1/d^3), \tag{4.36}$$

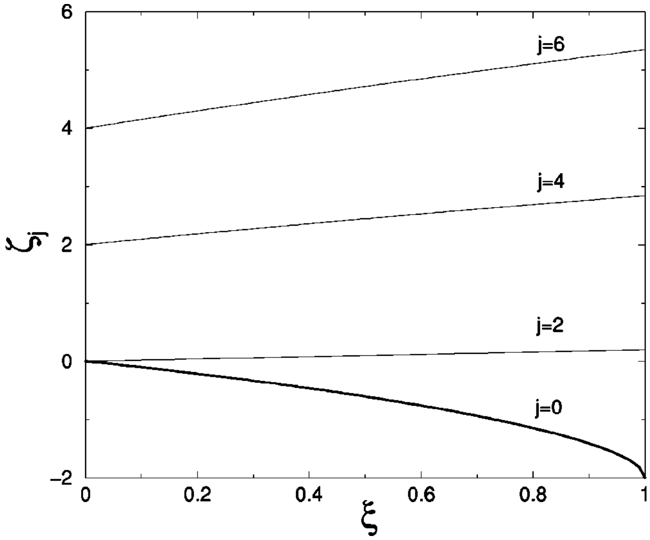


FIG. 1. Behavior of zero-mode exponents ζ_j ($j=0$, 2, 4, and 6) vs ξ for $d=3$. Notice the inequality $\zeta_0<\zeta_2<\zeta_4<\ldots$, which means the validity of the hierarchy (4.22) and thus the self-consistency of our zero-mode solutions.

where

$$K=(d-1)(d^3+2d^2\xi+3d^2+\xi^2 d+2d\xi-4+4\xi-\xi^2).$$

We stress that, for $j>2$, exponents $\zeta_j\equiv\zeta_j^-$ become admissible under the conditions already discussed in Sec. IV A.

The last remark concerns the self-consistency of our solution, that is, the validity of the hierarchy in (4.22). The validity of the latter can be easily verified from Fig. 1 where the behavior of ζ_j ($j=0$, 2, 4 and 6) is shown for $d=3$ as a function of ξ. Similar behaviors actually hold for all d's and j's. As we shall see in Sec. VI, a hierarchical order for the scaling exponents is also present for higher-order correlation functions.

V. UV RENORMALIZATION OF THE MODEL: RG FUNCTIONS AND RG EQUATIONS

The RG approach to the statistical models of the turbulence is exposed in Refs. [33, 34] in detail (see also Ref. [26] for the scalar Kraichnan model); below we confine ourselves to the only information we need.

The analysis of UV divergences is based on the analysis of canonical dimensions (see, e.g., Ref. [32]). Dynamical models of the type (3.1), in contrast to static models, have two scales, i.e., the canonical (''engineering'') dimension of some quantity F (a field or a parameter in the action functional) is described by two numbers, the momentum dimension d_F^k and the frequency dimension d_F^ω. They are determined so that $[F]\sim[\ell]^{-d_F^k}[T]^{-d_F^\omega}$, where ℓ is the length scale and T is the time scale. The dimensions are found from the obvious normalization conditions

$$d_k^k=-d_x^k=1,\ \ d_k^\omega=d_x^\omega=0,\ \ d_\omega^k=d_t^k=0,\ \ d_\omega^\omega=-d_t^\omega=1,$$

and from the requirement that each term of the action functional be dimensionless (with respect to the momentum and frequency dimensions separately). Then, based on d_F^k and d_F^ω, one can introduce the total canonical dimension $d_F=d_F^k+2d_F^\omega$ (in the free theory, $\partial_t\propto\partial^2$), which plays in the theory of renormalization of dynamical models the same role as the conventional (momentum) dimension does in static problems.

In the action (3.1), there are fewer terms than fields and parameters, and the canonical dimensions are not determined unambiguously. This is of course a manifestation of the fact that the ''superfluous'' parameter $B^o\equiv|\mathbf{B}^o|$ can be scaled out from the action (see Sec. III). After it has been eliminated (or, equivalently, zero canonical dimensions have been assigned to it), the definite canonical dimensions can be assigned to the other quantities. They are given in Table I, including the dimensions of renormalized parameters, which will appear later on. From Table I it follows that the model becomes logarithmic (the coupling constant g_0 becomes dimensionless) at $\xi=0$, and the UV divergences have the form of the poles in ξ in the Green functions.

The total canonical dimension of an arbitrary 1-irreducible Green function $\Gamma=\langle\Phi\ldots\Phi\rangle_{1-ir}$ is given by the relation

$$d_\Gamma=d_\Gamma^k+2d_\Gamma^\omega=d+2-N_\Phi d_\Phi, \tag{5.1}$$

where $N_\Phi=\{N_{\mathbf{B}'},N_{\mathbf{B}},N_{\mathbf{v}}\}$ are the numbers of corresponding fields $\Phi\equiv\{\mathbf{B}',\mathbf{B},\mathbf{v}\}$ entering into the function Γ, and the summation over all types of the fields is implied. The total dimension d_Γ is the formal index of the UV divergence. This means that superficial UV divergences, whose removal requires counterterms, can be present only in those functions Γ for which d_Γ is a non-negative integer.

Analysis of divergences in the problem (3.1) should be based on the following auxiliary considerations:

(i) All the 1-irreducible Green functions with $N_{\mathbf{B}'}<N_{\mathbf{B}}$ vanish (see Sec. III).

(ii) If for some reason a number of external momenta occur as an overall factor in all the diagrams of a given Green function, the real index of divergence d'_Γ is smaller than d_Γ by the corresponding number (the Green function requires counterterms only if d'_Γ is a non-negative integer).

In the model (3.1), the derivative $\boldsymbol{\partial}$ at the vertex can be moved onto the field $\mathbf{B}'$ using the integration by parts, which decreases the real index of divergence: $d'_\Gamma=d_\Gamma-N_{\mathbf{B}'}$. The field $\mathbf{B}'$ enters into the counterterms only in the form of a derivative, $\partial_\alpha B'_\beta$.

TABLE I. Canonical dimensions of the fields and parameters in the model (3.1).

F	$\mathbf{B},\mathbf{B}^o$	$\mathbf{B}'$	v	κ,κ_0	m, μ, Λ	D,D_0	g_0	g
d_F^k	0	d	-1	-2	1	$-2+\xi$	ξ	0
d_F^ω	0	0	1	1	0	1	0	0
d_F	0	d	1	0	1	ξ	ξ	0

(iii) A great deal of diagrams in the model (3.1) contain effectively closed circuits of retarded propagators $\langle \mathbf{BB}' \rangle_0$ and therefore vanish. For example, all the nontrivial diagrams of the 1-irreducible function $\langle B_\alpha B'_\beta v_\gamma \rangle_{1-ir}$ vanish.

From the dimensions in Table I we find $d_\Gamma = d + 2 - N_{\mathbf{v}} - dN_{\mathbf{B}'}$ and $d'_\Gamma = (d+2) - N_{\mathbf{v}} - (d+1)N_{\mathbf{B}'}$. From these expressions it follows that for any d, superficial divergences can only exist in the 1-irreducible functions with $N_{\mathbf{B}'} = 1$, $N_{\mathbf{v}} = N_{\mathbf{B}} = 0$ $(d_\Gamma = 2,\ d'_\Gamma = 1)$, $N_{\mathbf{B}'} = N_{\mathbf{B}} = 1$, $N_{\mathbf{v}} = 0$ $(d_\Gamma = 2,\ d'_\Gamma = 1)$, $N_{\mathbf{B}'} = N_{\mathbf{v}} = 1$, $N_{\mathbf{B}} = 0$ $(d_\Gamma = 1,\ d'_\Gamma = 0)$, and $N_{\mathbf{B}'} = N_{\mathbf{B}} = N_{\mathbf{v}} = 1$, $(d_\Gamma = 1,\ d'_\Gamma = 0)$ [we recall that $N_{\mathbf{B}} \leqslant N_{\mathbf{B}'}$ see (i) above]. However, the first of these counterterms has necessarily the form of a total derivative, $B^0_\alpha \partial^2 B_\alpha$, vanishes after the integration over **x** and therefore gives no contribution to the renormalized action. Furthermore, for the last two of these functions, all the nontrivial diagrams vanish [see (iii) above]. As in the case of the passive scalar field [26], we are left with the only superficially divergent function $\langle B'_\alpha B_\beta \rangle_{1-ir}$; the corresponding counterterm necessarily contains the derivative ∂ and therefore reduces to $B'_\alpha \partial^2 B_\alpha$ (another structure, $B'_\alpha \partial_\alpha \partial_\beta B_\beta$, vanishes by virtue of the solenoidality of **B**).

Introduction of this counterterm is reproduced by the multiplicative renormalization of the parameters g_0, κ_0 in the action functional (3.1) with the only independent renormalization constant Z_κ:

$$\kappa_0 = \kappa Z_\kappa, \quad g_0 = g\mu^\xi Z_g, \quad Z_g = Z_\kappa^{-1}. \tag{5.2}$$

Here μ is the reference mass in the minimal substraction scheme (MS), which we always use in what follows, g and κ are renormalized analogs of the bare parameters g_0 and κ_0, and $Z = Z(g,\xi,d)$ are the renormalization constants. Their relation in Eq. (5.2) results from the absence of renormalization of the contribution with K^{-1} in Eq. (3.1), so that $D_0 \equiv g_0\kappa_0 = g\mu^\xi\kappa$. No renormalization of the fields and the "mass" m is required, i.e., $Z_\Phi = 1$ for all Φ and $m_0 = m$. The renormalized action functional has the form

$$S_R(\Phi) = \mathbf{B}'[-\partial_t \mathbf{B} - (\mathbf{v}\cdot\boldsymbol{\partial})\mathbf{B} + (\mathbf{B}\cdot\boldsymbol{\partial})\mathbf{v} + (\mathbf{B}^o\cdot\boldsymbol{\partial})\mathbf{v} + Z_\kappa \kappa \partial^2 \mathbf{B}] - vK^{-1}\mathbf{v}/2, \tag{5.3}$$

where the function K from Eq. (2.2b) is expressed in renormalized parameters using Eqs. (5.2): $D_0 = g_0\kappa_0 = g\mu^\xi\kappa$.

The relation $S(\Phi, e_0) = S_R(\Phi, e, \mu)$ (where $e_0 = \{g_0, \kappa_0, m\}$ is the complete set of bare parameters, and $e = \{g, \kappa, m\}$ is the set of renormalized parameters) implies $W(e_0) = W_R(e,\mu)$ for the bare correlation functions $W = \langle \Phi \dots \Phi \rangle$ and their renormalized analogs W_R. We use $\widetilde{\mathcal{D}}_\mu$ to denote the differential operation $\mu\partial_\mu$ for fixed e_0 and operate on both sides of this equation with it. This gives the basic RG differential equation:

$$\mathcal{D}_{RG} W_R(e,\mu) = 0, \tag{5.4}$$

where $\mathcal{D}_{RG}$ is the operation $\widetilde{\mathcal{D}}_\mu$ expressed in the renormalized variables:

$$\mathcal{D}_{RG} \equiv \mathcal{D}_\mu + \beta(g)\partial_g - \gamma_\kappa(g)\mathcal{D}_\kappa. \tag{5.5}$$

In Eq. (5.5), we have written $\mathcal{D}_x \equiv x\partial_x$ for any variable x, and the RG functions (the β function and the anomalous dimension γ) are defined as

$$\gamma_F(g) \equiv \widetilde{\mathcal{D}}_\mu \ln Z_F \quad \text{for any } Z_F, \tag{5.6a}$$

$$\beta(g) \equiv \widetilde{\mathcal{D}}_\mu g = g[-\xi + \gamma_\kappa(g)]. \tag{5.6b}$$

The relation between β and γ in Eq. (5.6b) results from the definitions and the last relation in Eq. (5.2).

Now let us turn to the explicit calculation of the constant Z_κ in the one-loop approximation in the MS scheme. It is determined by the requirement that the 1-irreducible function $\langle \mathbf{B}'\mathbf{B} \rangle_{1-ir}$ expressed in renormalized variables be UV finite (i.e., be finite for $\xi \to 0$). This requirement determines Z_κ up to an UV finite contribution; the latter is fixed by the choice of a renormalization scheme. In the MS scheme all renormalization constants have the form "1 + only poles in ξ." The function $\langle \mathbf{B}'\mathbf{B} \rangle_{1-ir}$ in our model is known exactly [see Eqs. (3.4a) and (3.11)]. Let us substitute Eqs. (5.2) into Eqs. (3.4a), (3.11) and choose Z_κ to cancel the pole in ξ in the integral $J(m)$. This gives

$$Z_\kappa = 1 - gC_d \frac{(d-1)}{2d\xi}, \tag{5.7}$$

with the coefficient C_d from Eq. (3.11b). Note that the result (5.7) is exact, i.e., it has no corrections of order g^2, g^3, and so on; this is a consequence of the fact that the one-loop approximation (3.11) for the response function is exact. Note also that Eq. (5.7) coincides literally with the exact expression for Z_κ in the case of a passive scalar (see Ref. [26]).

For the anomalous dimension $\gamma_\kappa(g) \equiv \widetilde{\mathcal{D}}_\mu \ln Z_\kappa = \beta(g)\partial_g \ln Z_\kappa$ from the relations (5.6b) and (5.7) one obtains

$$\gamma_\kappa(g) = \frac{-\xi\mathcal{D}_g \ln Z_\kappa}{1 - \mathcal{D}_g \ln Z_\kappa} = gC_d \frac{(d-1)}{2d}. \tag{5.8}$$

From Eq. (5.6b) it then follows that the RG equations of the model have an IR stable fixed point $[\beta(g_*) = 0,\ \beta'(g_*) > 0]$ with the coordinate

$$g_* = \frac{2d\xi}{C_d(d-1)}. \tag{5.9}$$

Let $F(r)$ be some equal-time two-point quantity, for example, the pair correlation function of the primary fields $\Phi \equiv \{\mathbf{B}', \mathbf{B}, \mathbf{v}\}$ or some composite operators. We assume that $F(r)$ is multiplicatively renormalizable, i.e., $F = Z_F F^R$ with certain renormalization constant Z_F. The existence of nontrivial IR stable fixed point implies that in the IR asymptotic region $\Lambda r \gg 1$ and any fixed mr the function $F(r)$ takes on the form

$$F(r) \simeq \kappa_0^{d_F^\omega} \Lambda^{d_F} (\Lambda r)^{-\Delta_F} \chi(mr), \tag{5.10}$$

where d_F^ω and d_F are the frequency and total canonical dimensions of F, respectively, and χ is some function whose explicit form is not determined by the RG equation itself. The critical dimension Δ_F is given by the expression

$$\Delta[F] \equiv \Delta_F = d_F^k + \Delta_\omega d_F^\omega + \gamma_F^*, \tag{5.11}$$

where γ_F^* is the value of the anomalous dimension (5.6a) at the fixed point and $\Delta_\omega = 2 - \gamma_\kappa^* = 2 - \xi$ is the critical dimension of frequency [note that the value of $\gamma_\kappa(g)$ at the fixed point is also found exactly from the last relation in Eq. (5.6b): $\gamma_\kappa^* \equiv \gamma_\kappa(g_*) = \xi$].

The critical dimensions of the basic fields Φ in our model are found exactly [we recall that they are not renormalized and therefore $\gamma_\Phi = 0$ for all Φ]. From the dimensions in Table I we then find $\Delta_{\mathbf{v}} = 1 - \xi$, $\Delta_\theta = 0$, $\Delta_{\theta'} = d$.

VI. CRITICAL DIMENSIONS OF COMPOSITE OPERATORS

Any local (unless stated to be otherwise) monomial or polynomial constructed of primary fields and their derivatives at a single spacetime point $x \equiv \{t, \mathbf{x}\}$ is termed a composite operator. Examples are $\mathbf{B}^2$, $B'_\alpha \partial^2 B_\beta$, $B'_\alpha \mathbf{v}.\cdot \partial B_\alpha$, and so on.

Since the arguments of the fields coincide, correlation functions with these operators contain additional UV divergences, which are removed by additional renormalization procedure (see, e.g., Ref. [32]). For the renormalized correlation functions standard RG equations are obtained, which describe IR scaling with definite critical dimensions $\Delta_F \equiv \Delta[F]$ of certain "basis" operators F. Owing to the renormalization, $\Delta[F]$ does not coincide in general with the naive sum of critical dimensions of the fields and derivatives entering into F.

In general, composite operators are mixed in renormalization, i.e., a UV finite renormalized operator F^R has the form $F^R = F +$ counterterms, where the contribution of the counterterms is a linear combination of F itself and, possibly, other unrenormalized operators which "admix" to F in renormalization.

Let $F \equiv \{F_a\}$ be a closed set, all of whose monomials mix only with each other in renormalization. The renormalization matrix $Z_F \equiv \{Z_{ab}\}$ and the matrix of anomalous dimensions $\gamma_F \equiv \{\gamma_{ab}\}$ for this set are given by

$$F_a = \sum_b Z_{ab} F_b^F, \quad \gamma_F = Z_F^{-1} \mathcal{D}_\mu Z_F, \tag{6.1}$$

and the corresponding matrix of critical dimensions $\Delta_F \equiv \{\Delta_{ab}\}$ is given by Eq. (5.11), in which d_F^k, d_F^ω, and d_F are understood as the diagonal matrices of canonical dimensions of the operators in question (with the diagonal elements equal to sums of corresponding dimensions of all fields and derivatives constituting F) and $\gamma_F^* \equiv \gamma_F(g_*)$ is the matrix (6.1) at the fixed point (5.9).

Critical dimensions of the set $F \equiv \{F_a\}$ are given by the eigenvalues of the matrix Δ_F. The "basis" operators that possess definite critical dimensions have the form

$$F_a^{bas} = \sum_b U_{ab} F_b^R, \tag{6.2}$$

where the matrix $U_F = \{U_{ab}\}$ is such that $\Delta'_F = U_F \Delta_F U_F^{-1}$ is diagonal.

In general, counterterms to a given operator F are determined by all possible 1-irreducible Green functions with one operator F and arbitrary number of primary fields, $\Gamma = \langle F(x)\Phi(x_1)\cdots\Phi(x_n)\rangle_{1-ir}$. The total canonical dimension (formal index of divergence) for such functions is given by

$$d_\Gamma = d_F - N_\Phi d_\Phi, \tag{6.3}$$

with the summation over all types of fields $\Phi \equiv \{\mathbf{B}', \mathbf{B}, \mathbf{v}\}$ entering into the function. For superficially divergent diagrams, the real index of divergence, $d'_\Gamma = d_\Gamma - N_{\mathbf{B}'}$, is a nonnegative integer, cf. Sec. V.

In what follows, an important role will be played by the tensor composite operators built solely of the field $\mathbf{B}$ without derivatives:

$$F^{(np)}_{\alpha_1 \cdots \alpha_p}(x) \equiv B_{\alpha_1}(x) \cdots B_{\alpha_p}(x) [B_\alpha(x) B_\alpha(x)]^l, \quad n \equiv 2l + p. \tag{6.4}$$

Here p is the rank of the tensor and $n = 2l + p$ is the total number of fields $\mathbf{B}$ entering into the operator. From Table I and Eq. (6.3) for the operators (6.4) we obtain $d_F = 0$ and $d'_\Gamma = -N_{\mathbf{v}} - (d+1) N_{\mathbf{B}'}$. Therefore, the divergences can exist only in the functions with $N_{\mathbf{v}} = N_{\mathbf{B}'} = 0$, for which $d_\Gamma = d'_\Gamma = 0$. This means that the operators $F^{(np)}$ mix only with each other, i.e., the set (6.4) is closed with respect to the renormalization.

The simple analysis of the diagrams shows that the 1-irreducible function

$$\langle F^{(np)}(x) \mathbf{B}(x_1) \cdots B(x_{n'}) \rangle_{1-ir} \tag{6.5}$$

contains the factor $(B^o)^{n-n'}$ and therefore vanishes for $n' > n$, cf. the discussion in the end of Sec. III. It then follows that the operator $F^{(n'p')}$ can admix to $F^{(np)}$ only if $n' \leqslant n$. This means that the corresponding infinite renormalization matrix

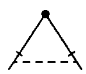
$$F^{(np)} = \sum_{n'p'} Z_{np,n'p'} F_R^{(n'p')} \tag{6.6}$$

is in fact block-triangular, i.e., $Z_{np,n'p'} = 0$ for $n' > n$, and so are the matrices γ_F, Δ_F, and U_F. It is then obvious that the critical dimensions associated with the operators $F^{(np)}$ are completely determined by the eigenvalues of the finite subblocks with $n' = n$. In the following, we shall not be interested in the precise form of the basis operators (6.2), we rather shall be interested in the anomalous dimensions themselves. Therefore, we can neglect all the elements of the matrix (6.6) other than $Z_{np,np'}$. The latter are found from the functions (6.5) which are independent of $\mathbf{B}^o$ and therefore can be calculated directly in the isotropic theory with $\mathbf{B}^o = 0$. It is then clear that the block $Z_{np,np'}$ can be diagonalized by the changing to irreducible operators: scalars ($p = 0$), vectors ($p = 1$), and traceless tensors ($p \geqslant 2$), but for our purposes it is sufficient to note that the elements of the block $Z_{np,np'}$ vanish for $p < p'$, i.e., this block is triangular along with the corresponding blocks of the matrices γ_F, Δ_F, and U_F. Indeed, the irreducible tensor of the rank p consists of the monomials with $p' \leqslant p$ only, for example, $F^{(22)}_{\alpha\beta} = B_\alpha B_\beta - \delta_{\alpha\beta} \mathbf{B}^2/d$, and therefore only these monomials can admix to the monomial of the rank p in renormalization. The final conclusion is that the critical dimensions, associated with the set (6.4), coincide with the diagonal elements $\Delta_{n,p} \equiv \Delta_{np,np}$ of the matrix (5.11), they are completely determined by the diagonal elements $Z_{np} \equiv Z_{np,np}$ of the matrix (6.6), and that they can be calculated directly in the isotropic theory with $\mathbf{B}^o = 0$.

Now let us turn to the one-loop calculation of the diagonal element Z_{np} of the matrix Z_F in the MS scheme. Let $\Gamma(x;\mathbf{B})$ be the generating functional of the 1-irreducible Green functions with one composite operator $F^{(np)}$ and any number of fields $\mathbf{B}$. Here $x \equiv \{t,\mathbf{x}\}$ is the argument of the operator and $\mathbf{B}(x)$ is the functional argument, the ''analog'' of the random field $\mathbf{B}(x)$. We are interested in the nth term of the expansion of $\Gamma(x;\mathbf{B})$ in $\mathbf{B}(\mathbf{x})$, which we denote $\Gamma^{(n)}(x;\mathbf{B})$; it has the form

$$\Gamma^{(n)}_{\alpha_1\cdots\alpha_p}(x;\mathbf{B}) = \frac{1}{n!}\sum_{\beta_1\cdots\beta_\alpha} \int dx_1 \cdots \int dx_n B_{\beta_1}(x_1)\cdots B_{\beta_n}(x_n) \times \langle F^{(np)}_{\alpha_1\cdots\alpha_p}(x) B_{\beta_1}(x_1)\cdots B_{\beta_n}(x_n)\rangle_{1-ir}. \tag{6.7}$$

In the one-loop approximation the functional (6.7) is represented diagrammatically as follows:

$$\Gamma^{(n)}_{\alpha_1\ldots\alpha_p} = F^{(np)}_{\alpha_1\ldots\alpha_p} + \frac{1}{2}$$  . (6.8)

Here the solid lines denote the bare propagators $\langle \mathbf{BB}\rangle_0$ from Eq. (3.3), the ends with a slash correspond to the field $\mathbf{B}'$, and the ends without a slash correspond to $\mathbf{B}$; the dashed line denotes the velocity propagator (2.2); the vertices correspond to the factor (3.2). The first term in Eq. (6.8) is the ''tree'' approximation, and the black circle with two attached lines in the diagram denotes the variational derivative

$$V_{\alpha_1\cdots\alpha_p\beta_1\beta_2}(x;x_1,x_2) \equiv \frac{\delta^2 F^{(np)}_{\alpha_1\cdots\alpha_p}(x)}{\delta B_{\beta_1}(x_1)\,\delta B_{\beta_2}(x_2)}. \tag{6.9}$$

It is convenient to represent it in the form

$$V_{\alpha_1\cdots\alpha_p\beta_1\beta_2}(x;x_1,x_2) = \delta(x-x_1)\,\delta(x-x_2) \times \frac{\partial^2}{\partial b_{\beta_1}\partial b_{\beta_2}}[b_{\alpha_1}\cdots b_{\alpha_p}(b^2)^l] \tag{6.10}$$

where b_α is a constant vector, which after the differentiation is substituted with the field $B_\alpha(x)$.

The vertex (6.10) contains $(n-2)$ factors of $\mathbf{B}$. Two remaining ''tails'' $\mathbf{B}$ are attached to the lower vertices of the diagram in Eq. (6.8). We know that the UV divergent part of the diagram is proportional to n factors $\mathbf{B}$ without derivatives, so that we can omit the first term of the vertex $\mathbf{B}'[-(\mathbf{v}\cdot\boldsymbol{\partial})\mathbf{B} + (\mathbf{B}\cdot\boldsymbol{\partial})\mathbf{v}]$, or, equivalently, the first term in Eq. (3.2). Furthermore, we can set all the external momenta in the integrand equal to zero, and the UV divergent part of the diagram (6.8) takes on the form

$$b_{\beta_3}b_{\beta_4}\frac{\partial^2}{\partial b_{\beta_1}\partial b_{\beta_2}}[b_{\alpha_1}\cdots b_{\alpha_p}(b^2)^l]T_{\beta_1\beta_2\beta_3\beta_4}, \tag{6.11}$$

where we have denoted

$$T_{\beta_1\beta_2\beta_3\beta_4} = D_0\int\frac{d\omega}{2\pi}\int\frac{d\mathbf{q}}{(2\pi)^d}\frac{q_{\beta_3}q_{\beta_4}P_{\beta_1\beta_2}(\mathbf{q})}{q^{d+\xi}[\omega^2+\kappa_0^2q^4]}. \tag{6.12}$$

We recall that the integration over $\mathbf{q}$ should be cut off from below at $q=m$ (see Sec. II). In Eq. (6.12), we have to change to the renormalized variables using Eqs. (5.2); in our approximation this reduces to the substitution $g_0 \to g\mu^\xi$ and $\kappa_0 \to \kappa$. Then we perform the integration over ω and use the relations (3.10) and

$$\int d\mathbf{q} f(q)\frac{q_{\beta_1}q_{\beta_2}q_{\beta_3}q_{\beta_4}}{q^4} = \frac{\delta_{\beta_1\beta_2}\delta_{\beta_3\beta_4}+\delta_{\beta_1\beta_4}\delta_{\beta_2\beta_3}+\delta_{\beta_1\beta_3}\delta_{\beta_2\beta_4}}{d(d+2)}\int d\mathbf{q}\, f(q).$$

This gives

$$T_{\beta_1\beta_2\beta_3\beta_4} = \frac{g\mu^\xi J(m)}{2d(d+2)}[(d+1)\delta_{\beta_1\beta_2}\delta_{\beta_3\beta_4} - (\delta_{\beta_1\beta_4}\delta_{\beta_2\beta_3}+\delta_{\beta_1\beta_3}\delta_{\beta_2\beta_4})], \tag{6.13}$$

with the integral $J(m)$ defined in Eq. (3.11b).

Substituting Eq. (6.13) into Eq. (6.11) gives the desired expression for the divergent part of the diagram (6.8). It is sufficient to take into account only the terms proportional to the monomial $B_{\alpha_1}(x)\cdots B_{\alpha_p}(x)[B_\alpha(x)B_\alpha(x)]^l$ and neglect all the other terms, namely, those containing the factors of $\delta_{\alpha_1\alpha_2}$, etc. The latter determine nondiagonal elements of the matrix Z_F, which we are not interested in here. Finally we obtain

$$\Gamma^{(n)}_{\alpha_1\cdots\alpha_p} \simeq F^{(np)}_{\alpha_1\cdots\alpha_p}\left[1-\frac{g\mu^\xi J(m)Q_{np}}{4d(d+2)}\right]+\cdots, \tag{6.14}$$

where we have written

$$Q_{np} \equiv 2n(n-1)-(d+1)(n-p)(d+n+p-2) = 2p(p-1)-(d-1)(n-p)(d+n+p). \tag{6.15}$$

The dots in Eq. (6.14) stand for the $O(g^2)$ terms and the structures different from $F^{(np)}$, $\simeq$ denote the equality up to UV finite parts; we also recall that $n=p+2l$.

The constant Z_{np} is found from the requirement that the renormalized analog $\Gamma_n^R \equiv Z_{np}^{-1}\Gamma_n$ of the function (6.14) be UV finite (mind the minus sign in the exponent); along with the expression (3.11b) for the integral $J(m)$ and the MS scheme this gives

$$Z_{np}=1-\frac{gC_dQ_{np}}{4d(d+2)\xi}+O(g^2), \qquad (6.16)$$

with C_d from Eq. (3.11b). For the anomalous dimension $\gamma_{np}=\tilde{\mathcal{D}}_\mu \ln Z_{np}$ it then follows

$$\gamma_{np}(g)=\frac{gC_dQ_{np}}{4d(d+2)}+O(g^2). \qquad (6.17)$$

From Table I and Eqs. (5.9) and (5.11) for the corresponding critical dimension $\Delta_{n,p}=\gamma_{np}(g_*)$ we finally obtain

$$\Delta_{n,p}(g)=\frac{\xi Q_{np}}{2(d-1)(d+2)}+O(\xi^2), \qquad (6.18)$$

with the polynomial Q_{np} from Eq. (6.15).

The straightforward analysis of the expression (6.18) shows that for fixed n and any d, the dimension $\Delta_{n,p}$ decreases monotonically with p and reaches its minimum for the minimal possible value, i.e., $p=0$ if n is even and $p=1$ if n is odd:

$$\Delta_{n,p}>\Delta_{n,p'} \quad \text{if } p>p'. \qquad (6.19a)$$

Furthermore, this minimal value is negative and it decreases monotonically as n increases:

$$0>\Delta_{2k,0}>\Delta_{2k+1,1}>\Delta_{2k+2,0}. \qquad (6.19b)$$

Finally, we note that for any fixed p, the dimension (6.18) decreases monotonically as n increases:

$$\Delta_{n,p}>\Delta_{n',p} \quad \text{if } n<n'. \qquad (6.19c)$$

The inequalities (6.19) show that the critical dimensions of the tensor operators (6.4) exhibit a kind of hierarchy; in particular, the less is the rank, the more negative is the dimension and, as will be explained in Sec. VII, the more important is its contribution to the inertial-range behavior.

In the model of passive scalar advection by the rapid-change velocity field (2.2) in the presence of an imposed linear gradient, similar inequalities are satisfied by the critical dimensions of tensor operators of the type (6.4), but constructed of gradients of the scalar field (see Ref. [15]). In the order $O(\xi)$ their critical dimensions coincide exactly with (6.18), which is, however, an artifact of the one-loop approximation (see Sec. VII).

As already said above, the operators that possess definite critical dimensions (6.18) are not (6.4) themselves, but the basis operators related to the latter by the relations (6.1) and (6.2). In the isotropic case $(\mathbf{B}^o=0)$, the basis operator with the dimension $\Delta_{n,p}$ is a p-th rank traceless tensor constructed of all the monomials $F^{(n'p')}$ with $n'=n$ and $p'\leqslant p$. When the background field $\mathbf{B}^o$ is ''turned on,'' the admixture of the monomials with $n'<n$ and $p'>p$ becomes possible. The ''missing'' fields $\mathbf{B}$ in the monomials with $n'<n$ are substituted with the constant fields $\mathbf{B}^o$ [the total number of the fields $\mathbf{B}$ and $\mathbf{B}^o$ has to be equal n, owing to the linearity of the basic equation (2.1) in $\mathbf{B}$ and $\mathbf{B}^o$], while the ''superfluous'' indices of the monomials with $p'>p$ are contracted with the indices of $\mathbf{B}^o$, so that the basis operator remains a pth rank traceless tensor. And vice versa, the unrenormalized monomial $F^{(np)}$ from (6.4) is a linear combination of the basis operators (6.2) with respective dimensions $\Delta_{n',p'}$. The hierarchy relations (6.19) then show that the minimal dimension entering into $F^{(np)}$ is Δ_{n,p_n}, where p_n is the minimal possible value of p for a given n, i.e., $p_n=0$ if n is even and $p_n=1$ if n is odd.

VII. OPERATOR PRODUCT EXPANSION AND THE ANOMALOUS SCALING FOR THE CORRELATION FUNCTIONS

The representation (5.10) for any scaling function $\chi(mr)$ describes the behavior of the Green function for $\Lambda r\gg 1$ and any fixed value of mr. The inertial range corresponds to the additional condition that $mr\ll 1$. The form of the function $\chi(mr)$ is not determined by the RG equations themselves; in the theory of critical phenomena, its behavior for $mr\to 0$ is studied using the well-known Wilson operator product expansion (OPE); see, e.g., Ref. [32]. This technique is also applicable to the theory of turbulence; see, e.g., Refs. [33, 34].

According to the OPE, the equal-time product $F_1(x)F_2(x')$ of two renormalized composite operators at $\mathbf{x}\equiv(\mathbf{x}+\mathbf{x}')/2=\text{const}$ and $\mathbf{r}\equiv\mathbf{x}-\mathbf{x}'\to\mathbf{0}$ has the representation

$$F_1(x)F_2(x')=\sum_a C_a(\mathbf{r})F_a(t,\mathbf{x}), \qquad (7.1)$$

where the functions C_a are the Wilson coefficients regular in m^2 and F_a are, in general, all possible renormalized local composite operators allowed by symmetry; more precisely, the operators entering into the OPE are those which appear in the corresponding Taylor expansions, and also all possible operators that admix to them in renormalization. If these operators have additional vector indices, they are contracted with the corresponding indices of the coefficients C_a.

Without loss of generality it can be assumed that the expansion in Eq. (7.1) is made in basis operators (6.2) with definite critical dimensions Δ_a. The renormalized correlation function $\langle F_1(x)F_2(x')\rangle$ is obtained by averaging Eq. (7.1) with the weight $\exp S_R$, the quantities $\langle F_a\rangle$ appear on the right-hand side. Their asymptotic behavior for $m\to 0$ is found from the corresponding RG equations and has the form $\langle F_a\rangle\propto m^{\Delta_a}$. From the operator product expansion (7.1) we therefore find the following expression for the scaling function $\chi(mr)$ in the representation (5.10) for the correlation function $\langle F_1(x)F_2(x')\rangle$:

$$\chi(mr)=\sum_a A_a(mr)^{\Delta_a}, \qquad (7.2)$$

where the coefficients $A_a = A_a(mr)$ are regular in $(mr)^2$; they depend on ξ, d and, in our case, on the cosine $z \equiv \cos\theta = \hat{\mathbf{B}}^o \cdot \mathbf{r}/r$.

Consider for definiteness the equal-time pair correlation function of the operators (6.4); their vector indices will be omitted in order to simplify the notation. For the leading term in the asymptotic region $\Lambda r \gg 1$ from the general expression (5.10) we obtain

$$\langle F^{(np)}(x_1) F^{(n'p')}(x_2) \rangle = (\Lambda r)^{-\Delta_{n,p_n} - \Delta_{n',p_{n'}}} \chi_{np,n'p'}(mr), \tag{7.3}$$

with the dimension $\Delta_{n,p}$ from Eq. (6.18) and certain functions $\chi_{np,n'p'}(mr)$. We recall that the monomial (6.4) is a linear combination of basis operators possessing definite critical dimensions (6.18) with different values of the indices; we also recall that p_n is the minimal possible value of p for a given n, i.e., $p_n = 0$ if n is even and $p_n = 1$ if n is odd. In Eq. (7.3), only the leading contribution is displayed, which is determined by the minimal dimensions entering into the operators on the left-hand side (see the discussion in the end of Sec. VI).

The leading term of the Taylor expansion for the function (7.3) involves the operators $F^{(kl)}$ from (6.4) with $k = n + n'$ and $l \leqslant p + p'$; higher-order terms involve tensors of arbitrary rank, built of the field $\mathbf{B}$ and its derivatives. The decomposition in renormalized operators gives rise to all the tensors $F^{(kl)}$ with $k \leqslant n + n'$ and all possible values of p; the tensors with $l > p + p'$ appear owing to the renormalization of the higher-order terms with derivatives. Therefore, the desired asymptotic expression for the function $\chi_{np,n'p'}(mr)$ in Eq. (7.3) in the region $mr \ll 1$ has the form

$$\chi_{np,n',p'}(mr) = \sum_{k=0}^{n+n'} \sum_j A_{kj} (mr)^{\Delta_{k,j}} + \cdots, \tag{7.4}$$

where A_{kj} are coefficients dependent only on ξ, d, and $z \equiv \cos\theta$, and the second summation runs over all values of j, allowed for a given k. Some remarks are now in order.

The leading term of the inertial-range behavior ($mr \ll 1$) of the function $\chi_{np,n'p'}(mr)$ is obviously given by the contribution with the minimal dimension $\Delta_{k,j}$ entering into Eq. (7.4).

The dots in Eq. (7.4) stand for the contributions of the order $(mr)^{2+O(\xi)}$ and higher, which arise from the senior operators, for example, $\mathbf{B}\partial^2\mathbf{B}$ and so on.

The operators $F^{(kj)}$ with $k > n + n'$ (whose contributions would be more important) do not appear in Eq. (7.4), because they do not appear in the Taylor expansion of the function (7.3) and do not admix in renormalization to the terms of the Taylor expansion. In other words, the number of the fields $\mathbf{B}$ in the operator F_a entering into the right-hand sides of the expansions (7.1) can never exceed the total number of the fields $\mathbf{B}$ in their left-hand sides.

The expansion (7.4) is consistent with the Legendre polynomial decomposition of the type (4.2) or, in general, with the decomposition in irreducible representations of the rotation group, employed in Refs. [13,14,36]. This becomes especially clear if the left-hand side of Eq. (7.1) involves only scalar quantities. Then all vector indices of the mean values $\langle F_a \rangle$ in the right-hand side are contracted with the indices of the corresponding Wilson coefficients $C_a(\mathbf{r})$. As is explained in Sec. VI, the basis operator that possesses definite critical dimension $\Delta_{k,j}$ is a jth rank traceless tensor, so that its mean value is also a jth rank traceless tensor, built solely of the constant vector $\mathbf{B}^o$ and Kronecker δ symbols. It is then clear that its contraction with $C_a(\mathbf{r})$ gives rise to the jth order Legendre polynomial $P_j(z)$.

Now let us turn to the comparison of the nonperturbative results for the pair correlation function $C_{\alpha\beta}(\mathbf{r}) = \langle \mathbf{B}_\alpha \mathbf{B}_\beta \rangle$, obtained in Sec. IV using the zero-mode techniques, with the predictions of the RG and OPE, given above. To this end, we put $n = n' = p = p' = 1$ in Eqs. (7.3), (7.4). The isotropic shell ($j = 0$) in Eq. (7.4) is then represented by the trivial operator $F = 1$ ($k = 0$) with $\Delta_{0,0} = 0$ and the monomial $\mathbf{B}^2 \equiv B_\alpha B_\alpha$ ($k = 2$) with $\Delta_{2,0} = -\xi + O(\xi^2)$ [see Eqs. (6.15) and (6.18)]. The leading term of the small-*mr* behavior is given by the latter, so that we have to identify $\Delta_{2,0}$ with $\zeta_0 \equiv \zeta_0^+$ from Eq. (4.35).

It was mentioned in Sec. VI that in the one-loop approximation, dimensions (6.18) coincide with the critical dimensions of tensor operators of the type (6.4), but constructed of the scalar gradients. The above identification shows that this coincidence is confined to the order $O(\xi)$ even for the simplest dimension $\Delta_{2,0}$. For the scalar case, one has $\Delta_{2,0} = -\xi$ *exactly*, in agreement with the well-known exact solution for the two-point structure function obtained in [21], while in our case $\Delta_{2,0}$ is a nontrivial function of ξ.

At first sight, the first anisotropic correction is related to the term with $k = j = 1$ in Eq. (7.4), i.e., to the simplest operator $\mathbf{B}$. However, the mean value $\langle \mathbf{B}(x) \rangle$ vanishes and therefore gives no contribution to Eq. (7.4). Indeed, the analysis of the diagrams shows that $\langle \mathbf{B}(x) \rangle$ is obtained from the 1-irreducible function $\langle \mathbf{B}'(x) \rangle_{1-ir}$, which vanishes owing to the invariance of the model (3.1) with respect to the shift $\mathbf{B}' \to \mathbf{B}' + \text{const}$.

The leading anisotropic correction is therefore related to the term with $k = j = 2$, i.e., with the operator $B_\alpha B_\beta$. Its dimension $\Delta_{2,2} = 2\xi/(d-1)(d+2) + O(\xi^2)$ has to be identified with $\zeta_2 \equiv \zeta_2^-$ and is in agreement with Eq. (4.36).

We have thus established the agreement between the $O(\xi)$ results obtained using the RG and OPE, with the first terms of the expansions in ξ of the exact nonperturbative results obtained within the zero-mode techniques. Note that for the isotropic exponent, such agreement was mentioned earlier in Ref. [30] to the order $O(\xi^2)$.

The exact expressions (4.35), (4.36) can therefore be viewed as nonperturbative predictions for the critical dimensions of the operators $\mathbf{B}^2 \equiv B_\alpha B_\alpha$ and $B_\alpha B_\beta$, respectively. Similarly, the results (4.27) for the higher exponents $\zeta_j^\pm$ can be linked to certain composite operators with two fields $\mathbf{B}$ and j derivatives for ζ_j^+ and $(j-2)$ derivatives for ζ_j^-. We shall not dwell on this point here and only note that the exponents ζ_2^+ and ζ_4^- are indeed related to the second-rank and fourth-rank families of the irreducible operators built of two fields $\mathbf{B}$ and two derivatives, $\partial B \partial B$, with various arrangements of the vector indices.

As is explained above in Sec. IV, the exponents $\zeta_j^\pm$ for $j \geqslant 4$ do not appear in the inertial-range behavior of the pair correlation function. This is also easily understood within the OPE. The mean value of the jth rank irreducible operator with n fields $\mathbf{B}$ is a traceless jth rank tensor built of n vectors

$\mathbf{B}^o$ and Kronecker δ symbols. This follows from the linearity of the basic equation (2.1) in $\mathbf{B}$ and $\mathbf{B}^o$ [see also the discussion in Sec. III below Eqs. (3.3)]. However, nonvanishing tensors of this type do not exist if the number of vector indices exceeds the number of fields [the structures like $B^o_{\alpha_1}B^o_{\alpha_2}B^o_{\alpha_3}B^o_{\alpha_4}/(B^o)^2$ are forbidden because $\mathbf{B}^o$ appears in the bare propagators (3.3) only in the numerators].

It was noted in Sec. IV that these exponents will be activated when a fully anisotropic forcing term (i.e., projecting onto all Legendre polynomials) is added to the right-hand side of Eq. (2.1). Moreover, the above interpretation in terms of the OPE shows that they are relevant even for the original simple model (2.1).

Although the contributions with $j>n$ vanish in the mean value $\langle B_\alpha(x)B_\beta(x')\rangle$, they are present in the expansion (7.1) *without averaging* and therefore the exponents $\zeta_j^{\pm}$ can reveal themselves in other correlation functions that involve the product $B_\alpha(x)B_\beta(x')$. In particular, they are relevant for the asymptotic behavior of the functions $\langle B_\alpha(x)B_\beta(x')\Phi(x_1)\cdots\Phi(x_n)\rangle$ for $x\to x'$. Of course, these exponents also appear in the representations (5.10) if the correlation function $F(r)$ in the left-hand side involves the operators with $j>n$.

VIII. DISCUSSION AND CONCLUSIONS

The zero-mode and RG techniques have been exploited in a model of magnetohydrodynamics turbulence where the magnetic field is passively advected by a Gaussian velocity δ-correlated in time, in the presence of a constant background magnetic field that introduces a large-scale anisotropy. The basic equations of the model are Eqs. (2.1)–(2.4). We have shown that the correlation functions of the magnetic fluctuations exhibit inertial-range anomalous scaling. The explicit asymptotic expressions for the correlation functions of the magnetic field and their powers have been obtained. In the inertial range, the correlation functions are represented as superpositions of power laws with universal exponents and nonuniversal amplitudes. The anomalous exponents have been calculated both nonperturbatively (for the second-order correlation function) and perturbatively (for the second- and higher-order correlation functions), in the first order of the exponent ξ and in any space dimension d.

In the language of the zero-mode techniques, anomalous exponents are associated with scale invariant functions which are annihilated by the inertial operator $\mathcal{L}$ (remember that in defining $\mathcal{L}$ we neglected the molecular diffusivity κ_0): the so-called zero modes of the equations for the correlation functions. In the language of the RG, these exponents are determined by the critical dimensions of tensor composite operators built of the magnetic field without derivatives, Eq. (6.4), and exhibit a kind of hierarchy related to the degree of anisotropy: the less is the rank, the less is the dimension and, consequently, the more important is the contribution to the inertial-range behavior. The leading terms of the even (odd) structure functions are given by the scalar (vector) operators.

For the pair correlation function, the complete set of the exponents has been calculated nonperturbatively using the exact equation (3.14); they are given in Eqs. (4.27) together with the discussion of their admissibility.

The general expressions (5.10), (7.3) describe the behavior of the correlation functions for $\Lambda r\gg 1$, and any fixed $mr(m\equiv 1/L)$, where $\Lambda^{-1}\propto\eta$, η being the dissipative scale, and L is the integral scale of the problem; expressions (7.2), (7.4) correspond to the additional condition $mr\ll 1$ (inertial range). These results for the leading terms can be summarized as follows:

$$\langle B_\parallel^n(t,\mathbf{x})B_\parallel^q(t,\mathbf{x}')\rangle\propto(\Lambda r)^{-\Delta_{n,p_n}-\Delta_{q,p_q}}(mr)^{\Delta_{n+q,p_{n+q}}}\propto r^{\zeta^{n,p}},$$

$$r=|\mathbf{x}-\mathbf{x}'| \tag{8.1}$$

with $\Delta_{n,p}$ given by [see Eqs. (6.15) and (6.18)]

$$\Delta_{n,p}=\xi\frac{2p(p-1)-(d-1)(n-p)(d+n+p)}{2(d-1)(d+2)}+O(\xi^2). \tag{8.2}$$

Here $B_\parallel$ is some component of $\mathbf{B}$, e.g., its projection onto the direction $\mathbf{r}/r$ or $\mathbf{B}^o/B^o$, p_n is the minimal possible value of p for a given n (i.e., $p=0$ for n even and $p=1$ for n odd). The exponents $\zeta^{n,q}$ are expressed through the dimensions $\Delta_{n,p}$ as follows:

$$\zeta^{n,q}=\begin{cases}\Delta_{n+q,0}-\Delta_{n,0}-\Delta_{q,0}=-\dfrac{\xi nq}{(d+2)}+O(\xi^2) & \text{if } n,q \text{ are even,}\\[2ex] \Delta_{n+q,0}-\Delta_{n,1}-\Delta_{q,1}=-\dfrac{\xi(nq+d+1)}{(d+2)}+O(\xi^2) & \text{if } n,q \text{ are odd,}\\[2ex] \Delta_{n+q,1}-\Delta_{n,0}-\Delta_{q,1}=-\dfrac{\xi nq}{(d+2)}+O(\xi^2) & \text{if } n \text{ is even and } q \text{ is odd.}\end{cases} \tag{8.3}$$

In the presence of an anisotropic forcing, questions about isotropy restoration at small scales are naturally raised. In particular, an issue recently addressed concerns the behavior of the derivative skewness factor of the passive scalar at large Péclet number, Pe, in the presence of large-scale anisotropy, and, in a more general formulation, the effects of large-scale anisotropy on the inertial-range statistics of passively advected fields [5,6,15,17] and the velocity itself [9,13,14]. In the case of passive advection of a scalar field, both the real [5] and the numerical experiments [6,17] show that the derivative skewness remains $O(1)$ for very high Péclet, in disagreement with what could be expected on the basis of both dimensional argumentations and cascade ideas. It means that, contrary to K41 hypothesis, anisotropy present at large scales persists at small scales. For the velocity field, in the case of a homogeneous shear flow an equivalent result has been found for the vorticity, which keeps a constant value, independent of the Reynolds number [9].

Let us now briefly discuss the consequences of our results for anisotropic indicators in this problem. Since the equation (2.1) is not invariant with respect to the shift $\mathbf{B}\to\mathbf{B}+\text{const}$, we can use as the simplest measure of small-scale anisotropy the dimensionless ratios of the correlation functions of the field $\mathbf{B}$ without derivatives, e.g.,

$$R_n \equiv \frac{\langle B_\parallel^{n-1}(\mathbf{x}) B_\parallel(\mathbf{x}')\rangle}{\langle B_\parallel(\mathbf{x}) B_\parallel(\mathbf{x}')\rangle^{n/2}}. \tag{8.4}$$

From Eqs. (8.3) it then follows that in inertial range of scales we have

$$R_{2k+1} \propto (\Lambda r)^{-\Delta_{2k,0}} (mr)^{\Delta_{2k+1,1}-(2k+1)\Delta_{2,0}/2}, \tag{8.5a}$$

$$R_{2k+2} \propto (\Lambda r)^{-\Delta_{2k+1,1}} (mr)^{\Delta_{2k+2,0}-(k+1)\Delta_{2,0}}. \tag{8.5b}$$

Note that the ratios (8.5) depend on both scales of wave number Λ and m; the dependence on the former follows from the fact that the powers $B_\parallel^n$ have nontrivial anomalous dimensions. The dependence on the Péclet number, $\text{Pe} \equiv (\Lambda/m)^\xi$ can be estimated by replacing r with $\eta = 1/\Lambda$; see Ref. [11]. Using explicit $O(\xi)$ expressions for $\Delta_{n,p}$ we then obtain:

$$R_{2k+1} \propto \text{Pe}^{-(d+2-4k^2)/[2(d+2)]}, \tag{8.6a}$$

$$R_{2k+2} \propto \text{Pe}^{2k(k+1)/(d+2)}. \tag{8.6b}$$

Since the leading terms of the even functions (8.1) are determined by the exponents of the isotropic shell (i.e., those related to scalar RG operators), the inertial-range behavior of the even ratios (8.5b), (8.6b) is the same as in the isotropic model. This gives a quantitative support to the universality of anomalous exponents with respect to different classes of forcing. On the other hand, the odd quantities (8.5a), (8.6a) appear to be sensitive to the anisotropy: R_3 in (8.6a) slowly decreases for $\text{Pe}\to\infty$, while ratios R_{2k+1} with $k\geqslant 2$ increase with Pe. Moreover, general expressions (8.5a) contain large Λ dependent factors, which also prevent these functions from vanishing at $\text{Pe}\to\infty$. Notice the important difference between the isotropic and the anisotropic problem: in the former the (nonuniversal) constant of the inertial-range power laws of odd-order moments are zero by symmetry, while this is not the case in the anisotropic context. This implies that the (hierarchical) exponents for the odd-order moments appear solely in the anisotropic case. For a given odd order, the leading exponent is thus responsible for the observed scale-dependent normalized odd order ratios.

The picture outlined above seems rather general. Indeed it is compatible with that recently established for the NS turbulence [13,14] and for the scalar field passively advected either by the velocity of the type (2.2) (see Ref. [15]) or by a NS velocity in the two-dimensional inverse cascade regime (see Ref. [17]). For a passive scalar field advected by a rapidly changing velocity field such as (2.2), RG expressions for the dimensionless ratios $R_n \equiv S_n/S_2^{n/2}$, S_n being the nth order structure function of the scalar field, are given by the expressions (8.5) *without* Λ dependent factors and with the same exponents of mr (see Ref. [11] for S_3 and Ref. [15] for general n's and d's). So, for example, for $k=1$ the ratio $S_{2k+1}/S_2^{k+(1/2)}$ decreases down to the small scales, but much slower than it was expected on the basis of dimensional argumentations, while for $k>1$ it grows in agreement with the results for Ref. [17].

It should be emphasized, however, that the results obtained within the lowest-order approximations in ξ are reliable only for moderate n, because the actual expansion parameter in the Kraichnan model is $n\xi$ rather than ξ itself (see Ref. [26]). The analysis of the large n behavior requires some additional resummation of the ξ series, which remains an open problem. For the passive scalar case, the numerical experiments [17] and the instanton calculus [37] show that the exponents analogous to $\zeta^{n,q}$ in Eq. (8.1) tend to a finite limit as $n\to\infty$ (''saturation''). It is worth noting that the limiting expressions for $n\to\infty$ obtained in Refs. [37] diverge as $\xi\to 0$, thus signalling that small ξ and large n limits do not commute. The persistence of small-scale anisotropies and the intermittency saturation are both statistical signatures of quasidiscontinuities observed in the scalar field [17]. It is then reasonable to expect that the saturation of intermittency takes place also in the magnetic case where quasidiscontinuous structures in the magnetic field are likely to be present (see, e.g., [38,39]).

To conclude, let us compare briefly the situation for the passively advected fields with the case of weak acoustic turbulence, where the spectra can be obtained as solutions of the linear kinetic equations (see Refs. [40,41]). For weakly dispersive waves (e.g., with the dispersion law $\omega(k)\propto k^{1+\delta}$ with $\delta\ll 1$), the anisotropy introduced by the large-scale forcing enhances going down towards to the depth of the inertial range [40]. The hierarchy of the exponents related to the Legendre decomposition is opposite to that established below and in Refs. [13–17]: anisotropic corrections decrease slower for larger j's [40]. On the contrary, for the nondispersive waves ($\delta=0$) the hierarchy of the exponents is similar to that in our case, the anisotropic corrections decay faster and faster with j and the spectrum tends to become isotropic at small scales [41]. To the best of our knowledge, no information is available for the higher-order correlation functions for such models. One can thus conclude that turbulent systems can exhibit essentially different types of behavior with respect to the small-scale isotropy restoration.

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APPENDIX A: COEFFICIENTS IN THE EQUATIONS FOR THE MAGNETIC CORRELATION FUNCTIONS

We report hereafter the coefficients $a_i, b_i, \ldots, r_i$ appearing in Eqs. (4.6)–(4.9).

1. Coefficients in Eq. (4.6)

$$a_1 = d-1,$$

$$b_1 = (d-1)(d-1-\xi),$$

$$c_1 = d-1+\xi,$$

$$d_1 = (d-1+\xi)(2\xi-d-3),$$

$$e_1 = -\xi^3+3\xi^2+2\xi(d-2)+2d(1-d),$$

$$f_1 = -2d\xi,$$

$$g_1 = 2\xi(d-2+\xi),$$

$$j_1 = -\xi[\xi^2+\xi(d-2)-2d],$$

$$k_1 = -2d\xi,$$

$$l_1 = -2\xi(2-d-\xi),$$

$$m_1 = -\xi(2\xi^2-8\xi+8-2d),$$

$$n_1 = 2\xi,$$

$$o_1 = -\xi(\xi-2),$$

$$p_1 = -\xi(\xi-2)(\xi-4),$$

$$q_1 = \xi(\xi-2),$$

$$r_1 = \xi(\xi-2)(\xi-4).$$

2. Coefficients in Eq. (4.7)

$$a_2 = (d+\xi-1)[2-2\xi+\xi(\xi-1)],$$

$$b_2 = d-1,$$

$$c_2 = (d+\xi-1)(d-1)+2\xi,$$

$$d_2 = d+\xi-1,$$

$$e_2 = -(d+\xi-1)(d-1),$$

$$f_2 = \xi[\xi^2+\xi(2d-3)+d^2-3d],$$

$$g_2 = 2\xi[\xi^2+\xi(d-2)-d],$$

$$k_2 = d+\xi-1,$$

$$l_2 = (d+\xi-1)(\xi-2),$$

$$m_2 = -(d+\xi-1),$$

$$n_2 = -(d+\xi-1)(\xi-2).$$

3. Coefficients in Eq. (4.8)

$$a_3 = (2-\xi)(d+\xi-1),$$

$$b_3 = -\xi(d+\xi-2),$$

$$c_3 = d-1,$$

$$d_3 = \xi+(d-1)^2,$$

$$e_3 = d+\xi-1,$$

$$f_3 = -(d+\xi-1)(d+1),$$

$$g_3 = -d^2+2d-\xi d+4\xi-1-2\xi^2,$$

$$j_3 = -d\xi,$$

$$k_3 = \xi,$$

$$l_3 = \xi(d+\xi-2),$$

$$m_3 = -2\xi(\xi-2),$$

$$n_3 = 2\xi(\xi-2).$$

4. Coefficients in Eq. (4.9)

$$a_4 = 2(2-\xi)(d+\xi-1),$$

$$b_4 = d-1,$$

$$c_4 = (d+\xi-1)(d-1)+2\xi,$$

$$d_4 = d+\xi-1,$$

$$e_4 = -(d+\xi-1)(d-1+2\xi),$$

$$f_4 = -2\xi,$$

$$g_4 = 2\xi.$$

APPENDIX B: RELATIONS INVOLVING THE LEGENDRE POLYNOMIALS

From the well-known relations involving the Legendre polynomials $P_j(z)$ (see, e.g., Ref. [35]) the following decompositions for a function $F(r,z)=\Sigma_{j=0}^{\infty} P_j(z) f_j(r)$ hold:

$$z\partial_z F = \sum_{j=0}^{\infty} P_j\left[j f_j + (2j+1)\sum_{q=1}^{\infty} f_{2q+j}\right], \tag{B1}$$

$$z^2\partial_z F = \sum_{j=0}^{\infty} P_j\left[\frac{j(j-1)}{2j-1} f_{j-1} - \frac{(j+1)(j+2)}{2j+3} f_{j+1} + (2j+1)\sum_{q=0}^{\infty} f_{2q+j+1}\right], \tag{B2}$$

$$zF = \sum_{j=0}^{\infty} P_j\left[\frac{j}{2j-1} f_{j-1} + \frac{j+1}{2j+3} f_{j+1}\right], \tag{B3}$$

$$z^2F=\sum_{j=0}^{\infty} P_j\left[\frac{j(j-1)}{(2j-1)(2j-3)}f_{j-2}+\left(\frac{(j+1)^2}{(2j+3)(2j+1)}+\frac{j^2}{(2j-1)(2j+1)}\right)f_j+\frac{(j+2)(j+1)}{(2j+3)(2j+5)}f_{j+2}\right], \tag{B4}$$

$$(1-z^2)\partial_z^2F=\sum_{j=0}^{\infty} P_j\left[j(1-j)f_j+2(2j+1)\sum_{q=1}^{\infty} f_{2q+j}\right], \tag{B5}$$

$$\partial_zF=\sum_{j=0}^{\infty} P_j\left[(2j+1)\sum_{q=0}^{\infty} f_{2q+j+1}\right]. \tag{B6}$$

APPENDIX C: FULL SET OF EQUATIONS PROJECTED ON THE LEGENDRE POLYNOMIALS

Inserting Eqs. (4.12) and (4.13) into Eqs. (4.6)–(4.11) and exploiting the relations reported in Appendix B, the following equations follow from the orthogonality of Legendre polynomials:

$$\begin{aligned}&a_1r^2f_j''^{(1)}+b_1rf_j'^{(1)}+c_1\left[j(1-j)f_j^{(1)}+2(2j+1)\sum_{q=1}^{\infty} f_{2q+j}^{(1)}\right]+d_1\left[jf_j^{(1)}+(2j+1)\sum_{q=1}^{\infty} f_{2q+j}^{(1)}\right]+e_1f_j^{(1)}+f_1f_j'^{(2)}\\&+g_1\left[jf_j^{(2)}+(2j+1)\sum_{q=1}^{\infty} f_{2q+j}^{(2)}\right]+j_1f_j^{(2)}+k_1r\left[\frac{j}{2j-1}f_{j-1}'^{(3)}+\frac{j+1}{2j+3}f_{j+1}'^{(3)}\right]+n_1(2j+1)\sum_{q=0}^{\infty} f_{2q+j+1}^{(3)}\\&+l_1\left[\frac{j(j-1)}{2j-1}f_{j-1}^{(3)}-\frac{(j+1)(j+2)}{2j+3}f_{j+1}^{(3)}+(2j+1)\sum_{q=0}^{\infty} f_{2q+j+1}^{(3)}\right]+m_1\left[\frac{j}{2j-1}f_{j-1}^{(3)}+\frac{j+1}{2j+3}f_{j+1}^{(3)}\right]+o_1f_j^{(5)}\\&+p_1\left[\frac{j(j-1)}{(2j-1)(2j-3)}f_{j-2}^{(5)}+\left(\frac{(j+1)^2}{(2j+3)(2j+1)}+\frac{j^2}{(2j-1)(2j+1)}\right)f_j^{(5)}+\frac{(j+2)(j+1)}{(2j+3)(2j+5)}f_{j+2}^{(5)}\right]\\&=B^{o2}\left[q_1+r_1\left(\frac{2}{3}\delta_{j,2}+\frac{1}{3}\delta_{j,0}\right)\right],\end{aligned} \tag{C1}$$

$$\begin{aligned}&a_2f_j^{(1)}+b_2r^2f_j''^{(2)}+c_2rf_j'^{(2)}+d_2\left[j(1-j)f_j^{(2)}+2(2j+1)\sum_{q=1}^{\infty} f_{2q+j}^{(2)}\right]+e_2\left[jf_j^{(2)}+(2j+1)\sum_{q=1}^{\infty} f_{2q+j}^{(2)}\right]\\&+f_2f_j^{(2)}+g_2\left[\frac{j}{2j-1}f_{j-1}^{(3)}+\frac{j+1}{2j+3}f_{j+1}^{(3)}\right]+k_2f_j^{(5)}\\&+l_2\left[\frac{j(j-1)}{(2j-1)(2j-3)}f_{j-2}^{(5)}+\left(\frac{(j+1)^2}{(2j+3)(2j+1)}+\frac{j^2}{(2j-1)(2j+1)}\right)f_j^{(5)}+\frac{(j+2)(j+1)}{(2j+3)(2j+5)}f_{j+2}^{(5)}\right]\\&=B^{o2}\left[m_2+n_2\left(\frac{2}{3}\delta_{j,2}+\frac{1}{3}\delta_{j,0}\right)\right],\end{aligned} \tag{C2}$$

$$\begin{aligned}&a_3(2j+1)\sum_{q=0}^{\infty} f_{2q+j+1}^{(1)}+b_3(2j+1)\sum_{q=0}^{\infty} f_{2q+j+1}^{(2)}+c_3r^2f_j''^{(3)}+d_3rf_j'^{(3)}+e_3\left[j(1-j)f_j^{(3)}+2(2j+1)\sum_{q=1}^{\infty} f_{2q+j}^{(3)}\right]\\&+f_3\left[jf_j^{(3)}+(2j+1)\sum_{q=1}^{\infty} f_{2q+j}^{(3)}\right]+g_3f_j^{(3)}+j_3r\left[\frac{j}{2j-1}f_{j-1}'^{(5)}+\frac{j+1}{2j+3}f_{j+1}'^{(5)}\right]+k_3(2j+1)\sum_{q=0}^{\infty} f_{2q+j+1}^{(5)}\\&+l_3\left[\frac{j(j-1)}{2j-1}f_{j-1}^{(5)}-\frac{(j+1)(j+2)}{2j+3}f_{j+1}^{(5)}+(2j+1)\sum_{q=0}^{\infty} f_{2q+j+1}^{(5)}\right]+m_3\left[\frac{j}{2j-1}f_{j-1}^{(5)}+\frac{j+1}{2j+3}f_{j+1}^{(5)}\right]\\&=n_3B^{o2}\delta_{j,2},\end{aligned} \tag{C3}$$

$$\begin{aligned}&a_4(2j+1)\sum_{q=0}^{\infty} f_{2q+j+1}^{(3)}+b_4r^2f_j''^{(5)}+c_4rf_j'^{(5)}+d_4\left[j(1-j)f_j^{(5)}+2(2j+1)\sum_{q=1}^{\infty} f_{2q+j}^{(5)}\right]+c_4\left[jf_j^{(5)}+(2j+1)\sum_{q=1}^{\infty} f_{2q+j}^{(5)}\right]+f_4f_j^{(5)}\\&=g_4B^{o2}\delta_{j,0},\end{aligned} \tag{C4}$$

$$rf_j'^{(1)}+(d-1)f_j^{(1)}+rf_j'^{(2)}-\left[jf_j^{(2)}+(2j+1)\sum_{q=1}^{\infty} f_{2q+j}^{(2)}\right]+r\left[\frac{j}{2j-1}f_{j-1}'^{(3)}+\frac{j+1}{2j+3}f_{j+1}'^{(3)}\right]+(2j+1)\sum_{q=0}^{\infty} f_{2q+j+1}^{(3)}$$
$$-\left[\frac{j(j-1)}{2j-1}f_{j-1}^{(3)}-\frac{(j+1)(j+2)}{2j+3}f_{j+1}^{(3)}+(2j+1)\sum_{q=0}^{\infty} f_{2q+j+1}^{(3)}\right]-\left[\frac{j}{2j-1}f_{j-1}^{(3)}+\frac{j+1}{2j+3}f_{j+1}^{(3)}\right]=0, \tag{C5}$$

$$(2j+1)\sum_{q=0}^{\infty} f_{2q+j+1}^{(2)}+rf_j'^{(3)}+df_j^{(3)}+r\left[\frac{j}{2j-1}f_{j-1}'^{(5)}+\frac{j+1}{2j+3}f_{j+1}'^{(5)}\right]+(2j+1)\sum_{q=0}^{\infty} f_{2q+j+1}^{(5)}-\frac{j(j-1)}{2j-1}f_{j-1}^{(5)}$$
$$+\frac{(j+1)(j+2)}{2j+3}f_{j+1}^{(5)}-(2j+1)\sum_{q=0}^{\infty} f_{2q+j+1}^{(5)}=0. \tag{C6}$$

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